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PAUL WEIDLINGER, CONSULTING ENGINEER

777 THIRD AVENUE

NEW YORK, NEW YORK. 10017

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TECHNICAL REPORT NO. 4

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ABSTRACT

The plane strain problem of a step load moving with uniform superseismic velocity $V > c_p$ on the surface of a half-space is considered for an elastic-plastic material obeying the von Mises yield condition.

Using dimensional analysis the governing quasi-linear partial differential equations are converted into ordinary nonlinear differential ones which are solved numerically using a digital computer. To overcome computing difficulties asymptotic solutions are derived in the vicinity of a singular point of the differential equations.

Numerical results are presented for a range of selected values of the significant nondimensional parameters, i.e. of the surface load $\frac{p_o}{k}$, of Poisson's ratio ν and of the velocity ratio $\frac{V}{c_p}$.

TABLE OF CONTENTS

LIST OF SYMBOLS*)

$a_{1...4}$	Functions defined by Eqs. (A-8) to (A-15).
$b_{1...4}$	Functions defined by Eqs. (III-14)-(III-16) and (III-26).
c_P , c_S , $\bar{c}$	Velocity of propagation of elastic P-waves, S-waves and of inelastic shock fronts, respectively.
F	Plastic potential, Eq. (II-1).
G	Shear modulus.
J_1 , J_2'	Invariants of stress.
k	Yield stress in shear, Eq. (II-1).
$K = \frac{2(1+\nu)}{3(1-2\nu)} G$	Bulk modulus.
$L > 0$	Function related to inelastic behavior, Eq. (II-31).
$p(x - Vt)$	Surface pressure.
p_E , p_L , p_o	Intensities of step load surface pressure.
s_1 , s_2	Principal stress deviators.
s_x , s_y , s_{ij}	Stress deviators with respect to axes x, y, etc.
t	Time.
v_x , v_y	Components of particle velocity in x and y directions, respectively.
V	Velocity.
x, y	Cartesian coordinates, Fig. 1.
$X = \frac{\rho V^2}{2G} \sin^2 \varphi$	Nondimensional expression.
X_P , X_S	Values of X at P- and S-fronts, respectively.
$\beta = \frac{s_1 - s_2}{s_1 + s_2}$	Nondimensional stress variable.
γ	Angle between the directions of s_1 and of the position ray of an element, Fig. 3.

*) Other symbols, which are used in one location only, are defined as they occur.

$\Delta \equiv \beta - 3$	Small quantity for purposes of asymptotic expansion.
$\Delta\sigma$, Δv, $\Delta\tau$, etc.	Increments of σ, v, τ, etc. at a front.
$\epsilon \equiv \varphi - \bar{\varphi}$	Small quantity for purposes of asymptotic expansion.
ϵ_{ij} , $\dot{\epsilon}_{ij}$	Strains, strain-rates.
$\eta \equiv \gamma - \frac{\pi}{2}$	Small quantity for purposes of asymptotic expansion.
θ	Angle defining direction of the principal stress σ_1 , Fig. 3.
$\lambda > 0$	Function related to inelastic behavior, Eq. (II-8).
ν	Poisson's ratio.
$\xi \equiv \frac{x - Vt}{y}$	Variable defined by Eq. (II-16).
ρ	Mass density of medium.
σ_{ij} , $\dot{\sigma}_{ij}$	Stresses, stress rates.
σ_1 , σ_2 , σ_3	Principal stresses.
τ	Shear stress.
φ	Position angle of element, Fig. 4.
φ_P , φ_S , $\bar{\varphi}$	Position of the elastic P- and S- and of the inelastic shock fronts, respectively.
φ_1 , φ_2 , φ_3 , φ_4	Limits of inelastic regions.
$'$	Differentiation with respect to φ.

I INTRODUCTION.

The two dimensional problem of the effect of a pressure pulse $p(x - Vt)$ progressing with the velocity V on the surface of an elastic half-space, Fig. 1, has been treated by Cole and Huth [1] for a line load and, by superposition, may be found for any other distribution $p(x - Vt)$. Miles [2] has considered the three dimensional problem of loads with axially symmetric distribution $p(r,t)$ over an expanding circular area on the surface, Fig. 2. He has demonstrated that the plane problem [1] contains the asymptotic solution for the three dimensional problem [2] in the region near the wave front. The actual solution of the three dimensional problem would require a great numerical effort which can be avoided, by using the solution of the plane problem to estimate the effect of circularly expanding surface loads.

Real materials can not be expected to be elastic, and solutions of the three dimensional problem, Fig. 2, for dissipative materials are hopelessly complex. However, estimates for the three dimensional case can be made from generalizations of the problem treated in [1] for dissipative materials. This has been done for linearly viscoelastic materials by Sackman [3], and Workman and Bleich [4], in the superseismic and subseismic ranges, respectively. Based on a formal solution in [5] the effect of a step load moving with superseismic velocity is determined in the present report for an elastic-plastic material obeying the von Mises yield condition. The identical problem, for a yield condition suitable for materials with internal slip subject to Coulomb friction, is concurrently being treated for publication elsewhere [6].

The yield mechanism in the medium makes the problem nonlinear, such that superposition is not permitted and each pressure distribution $p(x - Vt)$ poses a separate problem. The present paper treats the case of a progressing step load

$p(x - Vt) = p_o H(Vt - x)$. An approach permitting an approximate solution of the important, but very complex case of a decaying surface pressure is discussed in the Conclusions.

The formulation of the problem in [5] furnishes a set of simultaneous, ordinary nonlinear differential equations which are solved numerically by a Runge-Kutta forward integration scheme [7] utilizing an IBM 7090. The integration encounters numerical difficulties near singularities of the system of differential equation. To treat such situations asymptotic expansions were employed, leading to approximate differential equations which could then be integrated numerically.

Numerical results for all combinations of the nondimensional parameters $\nu = 0, \frac{1}{8}, \frac{1}{4}, 0.35$ and $\frac{V}{c_P} = 1.25, 1.5, 2.0, 4.0$, are given in Tables 1-16 in each case for five values of the surface load $\frac{p_o}{k}$. The coverage of results is extensive enough to permit interpolation.

II FORMULATION OF THE BASIC EQUATIONS.

Figure 3 indicates the half-space and a system of Cartesian coordinates. The x-axis is in the direction of motion of the step load, the y- and z-axes are normal to the surface in and out of the plane of the figure, respectively. The analysis considers the case of plane strain, $e_z = 0$, when the velocity V of the step load is superseismic. It is known [8] that the largest wave velocity in an elastic-plastic material is c_p, the velocity of elastic P-waves in the material, so that the term superseismic means $V > c_p$. Throughout the analysis it is assumed that the strains and velocities are small, so that their higher powers may be neglected in comparison to linear terms.

To describe the behavior of the elastic-plastic material the plastic potential is introduced

$$F = J_2' - k^2 \qquad \text{(II-1)}$$

where J_2' is the invariant

$$J_2' = \frac{1}{2} s_{ij} s_{ji} \qquad \text{(II-2)}$$

and the value $k > 0$ is the yield stress in shear.

The behavior of an element of the material is then defined by the statements which follow.

1. The value of the function F may never be positive

$$F \leq 0 \qquad \text{(II-3)}$$

2. If, in an element of the material at a given instant,

$$F < 0 \qquad \text{(II-4)}$$

the rates of change in stress and strain are related by the conventional elastic relations.

3. However, if the yield condition

$$F = 0 \qquad \text{(II-5)}$$

is satisfied three possibilities exist: a) in the next instant of time the material may be in a state of plastic deformation; b) it may be in a state of elastic unloading; c) it may be in a neutral state.

a) If the material is in a state of plastic deformation

$$\dot{F} = 0 \qquad \text{(II-6)}$$

the total strain rate will be the sum of an elastic and a plastic portion

$$\dot{\epsilon}_{ij} = \dot{\epsilon}^{E}_{ij} + \dot{\epsilon}^{P}_{ij} \qquad \text{(II-7)}$$

where $\dot{\epsilon}^{E}_{ij}$ is obtained from the conventional elastic relations, while

$$\dot{\epsilon}^{P}_{ij} = \lambda \frac{\partial F}{\partial \sigma_{ij}} \qquad \text{(II-8)}$$

λ, which must be positive

$$\lambda > 0 \qquad \text{(II-9)}$$

is an a priori unknown function of space and time. It is to be found as part of the solution of the problem.

b) In case of elastic unloading $\dot{F} < 0$ holds and the elastic stress-strain relations apply.

c) In the neutral state $\dot{F}$ vanishes as in case a, but neither energy dissipation nor permanent deformation occurs and the elastic stress-strain relations apply. In the present problem neutral regions will be encountered in which neither the stress nor the strain changes, $\dot{\epsilon}_{ij} = \dot{\sigma}_{ij} = 0$.

For the purpose of this paper it is convenient to combine elastic and neutral regions, jointly to be called "nondissipative", as opposed to plastic regions, where $\lambda > 0$, indicating that energy is dissipated. In the nondissipative regions the changes in stress and strain are governed by the elastic relations, while in plastic regions, Eqs. (7) and (8) apply. Formally, the equations in nondissipative regions can therefore be obtained by substitution of $\lambda = 0$ into the differential equations derived below for the plastic regions and by replacing the conditions $F = \dot{F} = 0$ by the inequality (3).

Substituting Eqs. (7) and (8) and the elastic stress-strain relations into

$$\dot{\epsilon}_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) \qquad \text{(II-10)}$$

the following constitutive equations are obtained for the case of plane strain

$$\left.\begin{aligned}
&\frac{1}{2G}\dot{s}_x + \frac{1-2\nu}{6(1+\nu)G}\dot{J}_1 + \lambda s_x = \frac{\partial v_x}{\partial x} \\
&\frac{1}{2G}\dot{s}_y + \frac{1-2\nu}{6(1+\nu)G}\dot{J}_1 + \lambda s_y = \frac{\partial v_y}{\partial y} \\
&\frac{1}{2G}\dot{\tau} + \lambda\tau = \frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \\
&\frac{1}{2G}(\dot{s}_x + \dot{s}_y) - \frac{1-2\nu}{6(1+\nu)G}\dot{J}_1 + \lambda(s_x + s_y) = 0
\end{aligned}\right\} \qquad \text{(II-11)}$$

J_1 is the first invariant of stress, s_x , s_y and v_x , v_y are, respectively, the stress deviators, and particle velocity components in the x- and y-directions.

Further, there are two equations of motion

$$\left.\begin{aligned}\frac{\partial s_x}{\partial x} + \frac{1}{3}\frac{\partial J_1}{\partial x} + \frac{\partial \tau}{\partial y} &= \rho \frac{\partial v_x}{\partial t}\\ \frac{\partial s_y}{\partial y} + \frac{1}{3}\frac{\partial J_1}{\partial y} + \frac{\partial \tau}{\partial x} &= \rho \frac{\partial v_y}{\partial t}\end{aligned}\right\} \qquad \text{(II-12)}$$

Equations (11) and (12) and the respective requirements on F and λ complete the formulation except for initial and boundary conditions.

In plastic regions additional equation $F = 0$ applies, so that there are a total of seven relations for the seven unknown quantities s_x, s_y, τ, J_1, v_x, v_y and $\lambda > 0$. In nondissipative regions Eqs. (11), (12) apply, but the function λ vanishes identically, $\lambda \equiv 0$, while F must satisfy either $F < 0$, or the two conditions $F = 0$, $\dot{F} \leq 0$ simultaneously. In the nondissipative case there are only six differential equations and six unknown quantities. The complete solution of the problem is to be obtained from the six differential equations (11), (12) and the applicable relations on λ and F, subject to appropriate boundary or initial conditions at the surface and at the junctions of the as yet unknown regions:

1. On the surface, $y = 0$, a step pressure $p = p_o H(Vt - x)$ normal to the surface is applied, so that,

$$\sigma_y = \begin{cases} -\,p_o & (\text{for } Vt \geq X)\\ 0 & (\text{for } Vt < X)\end{cases} \qquad \text{(II-13)}$$

while

$$\tau \equiv 0 \qquad \text{(II-14)}$$

2. It is known from a general study of elastic-plastic wave propagation [8] that the largest characteristic velocity possible is c_p , the velocity of elastic P-waves. All stresses and velocities must therefore vanish outside the wedge formed by the loaded portion of the surface and the P-front, which is inclined at the angle

$$\varphi_p = \pi - \sin^{-1}\left[\frac{1}{V}\sqrt{\frac{2G(1-\nu)}{\rho(1-2\nu)}}\right] \tag{II-15}$$

with the x-axis, Fig. 4.

Because the steady-state problem is considered, stresses and velocities can not be functions of x and t separately, but must be of the form $f(x - Vt)$. For the step load $p = p_o H(Vt - x)$ dimensional considerations discussed in detail in [5] require further that the stresses and velocities do not depend on $x - Vt$ and y separately, but must be solely functions of the combination

$$\xi = \frac{x - Vt}{y} \tag{II-16}$$

Instead of using ξ as variable, the equivalent but more convenient one

$$\varphi = \cot^{-1}\xi \tag{II-17}$$

is introduced. The angle φ is shown in Fig. 3.

Noting $\frac{d\xi}{d\varphi} = -\frac{1}{\sin^2\varphi}$, one obtains the relations

$$\frac{\partial}{\partial x} = \frac{1}{y}\frac{d}{d\xi} = -\frac{\sin^2\varphi}{y}\frac{d}{d\varphi} \tag{II-18}$$

$$\frac{\partial}{\partial y} = -\frac{\xi}{y}\frac{d}{d\xi} = \frac{\sin 2\varphi}{2y}\frac{d}{d\varphi} \tag{II-19}$$

$$\frac{\partial}{\partial t} = -\frac{V}{y}\frac{d}{d\xi} = \frac{V}{y}\sin^2\varphi\frac{d}{d\varphi} \tag{II-20}$$

Substitution into Eqs. (11), (12) reduces these equations to a set of ordinary simultaneous differential equations in the single independent variable φ.

Because of the manner of solution to be employed the unknowns s_x , s_y and τ are replaced by three other dependent variables, s_1 , s_2 and θ,

$$s_x = s_1 \cos^2 \theta + s_2 \sin^2 \theta \tag{II-21}$$

$$s_y = s_2 \cos^2 \theta + s_1 \sin^2 \theta \tag{II-22}$$

$$\tau = (s_1 - s_2) \sin \theta \cos \theta \tag{II-23}$$

s_1 and s_2 are the two principal stress deviators, while θ is in the angle between the direction of s_1 and the horizontal, Fig. 3. Introducing further the angle γ between the direction of s_1 and the position vector, Fig. 3,

$$\gamma = \varphi - \theta \tag{II-24}$$

the six differential equations, (11), (12) become finally

$$s_1' \cos^2 \theta + s_2' \sin^2 \theta - (s_1 - s_2)\, \theta' \sin 2\theta + \\ + \frac{1-2\nu}{3(1+\nu)} J_1' + L(s_1 \cos^2 \theta + s_2 \sin^2 \theta) = - \frac{2G}{V} v_x' \tag{II-25}$$

$$s_1' \sin^2 \theta + s_2' \cos^2 \theta + (s_1 - s_2)\, \theta' \sin 2\theta + \\ + \frac{1-2\nu}{3(1+\nu)} J_1' + L(s_1 \sin^2 \theta + s_2 \cos^2 \theta) = \frac{2G}{V} v_y' \cot \varphi \tag{II-26}$$

$$\frac{1-2\nu}{1+\nu} J_1' = \frac{2G}{V} (v_y' \cot \varphi - v_x') \tag{II-27}$$

$$(s_1' - s_2') \sin 2\theta + 2(s_1 - s_2)\, \theta' \cos 2\theta + L(s_1 - s_2) \sin 2\theta = \\ = \frac{G}{V} (v_x' \cot \varphi - v_y') \tag{II-28}$$

$$s_1' \cos \theta \sin \gamma + s_2' \sin \theta \cos \gamma - (s_1 - s_2)\,\theta' \cos (\gamma - \theta) +$$
$$+ \frac{1}{3} J_1' \sin \varphi = - \rho V v_x' \sin \varphi \qquad \text{(II-29)}$$

$$s_1' \sin \theta \sin \gamma - s_2' \cos \theta \cos \gamma + (s_1 - s_2)\,\theta' \sin (\gamma - \theta) -$$
$$- \frac{1}{3} J_1' \cos \varphi = - \rho V v_y' \sin \varphi \qquad \text{(II-30)}$$

Primes indicate differentiation with respect to φ and the function L is related to λ,

$$L = \frac{2G\gamma}{V \sin^2 \varphi} \lambda \qquad \text{(II-31)}$$

The function L is subject to the same conditions as λ, i.e. $L > 0$ in plastic regions, $L = 0$ elsewhere.

The expression for the plastic potential in these variables is

$$F = s_1^2 + s_1 s_2 + s_2^2 - k^2 \qquad \text{(II-32)}$$

In plastic regions Eq. (6) requires $\dot{F} = 0$, giving the additional differential equation

$$(2s_1 + s_2)\, s_1' + (s_1 + 2s_2)\, s_2' = 0 \qquad \text{(II-33)}$$

The unknowns v_x' and v_y' can be eliminated from Eqs. (25)-(30) without differentiation. Using the symbol

$$X = X(\varphi) = \frac{\rho V^2}{2G} \sin^2 \varphi \qquad \text{(II-34)}$$

the operations

Eq. (40a) = Eq. (25) + Eq. (26) - Eq. (27) (II-35)

Eq. (40b) = - X Eq. (27) + $\sin \varphi$ Eq. (29) - $\cos \varphi$ Eq. (30) (II-36)

$$\text{Eq. (40c)} = X\left\{\sin 2\theta\,[\text{Eq. (26)} - \text{Eq. (25)}] + \cos 2\theta\ \text{Eq. (28)}\right\} + \\ + \cos(\gamma - \theta)\ \text{Eq. (29)} - \sin(\gamma - \theta)\ \text{Eq. (30)} \tag{II-37}$$

$$\text{Eq. (40d)} = X\left\{\cos 2\theta\,[\text{Eq. (26)} - \text{Eq. (25)}] - \sin 2\theta\ \text{Eq. (28)}\right\} + \\ + \sin(\gamma - \theta)\ \text{Eq. (29)} - \cos(\gamma - \theta)\ \text{Eq. (30)} \tag{II-38}$$

$$\text{Eq. (40e)} = \text{Eq. (33)} \tag{II-39}$$

lead to the following set of five differential equations valid in plastic regions:

$$\begin{bmatrix} 1 & 1 & -\left(\frac{1-2\nu}{1+\nu}\right) & 0 & (s_1 + s_2) \\ \sin^2\gamma & \cos^2\gamma & 1-3X\left(\frac{1-2\nu}{1+\nu}\right) & -\sin 2\gamma & 0 \\ \frac{1}{2}\sin 2\gamma & \frac{1}{2}\sin 2\gamma & \sin 2\gamma & 2X-1 & 0 \\ \sin^2\gamma - X & X - \cos^2\gamma & -\cos 2\gamma & 0 & -X(s_1 - s_2) \\ 2s_1 + s_2 & s_1 + 2s_2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s_1' \\ s_2' \\ \frac{1}{3}J_1' \\ (s_1 - s_2)\theta' \\ L \end{bmatrix} = 0 \tag{II-40}$$

The system of equations for nondissipative regions consists of the first four Eqs. (40) without the terms containing L, i.e.

$$\begin{bmatrix} -1 & -1 & \frac{1-2\nu}{1+\nu} & 0 \\ \sin^2\gamma & \cos^2\gamma & 1-3X\,\frac{1-2\nu}{1+\nu} & -\sin 2\gamma \\ \sin 2\gamma & \sin 2\gamma & 2\sin 2\gamma & -2(1-2X) \\ \sin^2\gamma - X & X-\cos^2\gamma & -\cos 2\gamma & 0 \end{bmatrix} \begin{bmatrix} s_1' \\ s_2' \\ \frac{1}{3}J_1' \\ (s_1-s_2)\theta' \end{bmatrix} = 0 \tag{II-41}$$

III SOLUTIONS FOR INDIVIDUAL REGIONS.

As a first step towards the construction of overall solutions, expressions for individual regions must be derived. The latter will be combined in Section IV to find the solution for the entire domain.

a) Nondissipative Regions.

Equations (II-41) are linear and homogeneous so that the derivatives of the stresses s_1' , s_2' , J_1' and the value $(s_1 - s_2)\,\theta'$ vanish, unless the coefficient matrix in Eqs. (II-41) is singular, requiring

$$X(1-2X)\,[1 + (1-2X)(1-2\nu)] = 0 \tag{III-1}$$

Equation (1) has two significant roots,

$$X_P = \frac{1-\nu}{1-2\nu} \tag{III-2}$$

and

$$X_S = \frac{1}{2} \tag{III-3}$$

Substitution of the two roots X_P and X_S into (II-34) furnishes the two locations

$$\varphi_P = \pi - \sin^{-1} \frac{c_P}{V} \tag{III-4}$$

$$\varphi_S = \pi - \sin^{-1} \frac{c_S}{V} \tag{III-5}$$

where c_P , c_S are the velocities of P- and S-waves, respectively. In all locations $\varphi = \varphi_P$ or φ_S the values s_1' , s_2' , J_1' , $(s_1 - s_2)\,\theta'$ vanish, so that in nondissipative regions the stresses must remain constant except at the locations φ_P and φ_S . The latter being the potential locations of elastic P- and S-shock fronts, respectively, it is known that discontinuities in stresses and velocities may occur at these locations and may, therefore, be part of the complete solutions to be constructed. The following pertinent details will be required subsequently.

(1) The P-front.

Designating the discontinuous changes in the various quantities at the front by the symbol Δ, the discontinuities in the stresses σ_N , $\sigma_T = \sigma_z$ (normal and tangential to the front, respectively) and in the component v_N of the velocity (normal to the front) are proportional to $\Delta\sigma_N$.

$$\Delta\sigma_T = \frac{\nu}{1-\nu}\,\Delta\sigma_N \quad , \quad \Delta v_N = -\frac{\Delta\sigma_N}{\rho c_p} \tag{III-6}$$

No other discontinuities can occur in this location.

The changes $\Delta\sigma_N$ and $\Delta\sigma_T$ are of course limited by the yield condition $F \leq 0$ which must be satisfied on either side of the front.

In the actual solution a P-front will be encountered only when the region ahead of the front is stressless and at rest. The normal to the front is then a principal direction for the stresses behind the front, so that $\gamma = 0$ or $\frac{\pi}{2}$. Selecting $\sigma_1 = \sigma_N$, corresponds to

$$\gamma = \frac{\pi}{2} \tag{III-7}$$

The corresponding values of the other quantities of interest behind the shock front are

$$\beta = 3 \quad , \quad s_1 = \frac{2(1-2\nu)\sigma_1}{3(1-\nu)} \quad , \quad s_2 = -\frac{(1-2\nu)\sigma_1}{3(1-\nu)} \quad , \quad J_1 = \frac{(1+\nu)\sigma_1}{1-\nu} \tag{III-8}$$

subject to the limitation

$$|\sigma_1| \leq \frac{\sqrt{3}(1-\nu)k}{1-2\nu} \tag{III-9}$$

imposed by the yield condition.

(2) The S-front.

At an S-front discontinuities occur only in the shear stress $\tau_N = \tau_T = \tau$ and in the tangential velocity v_T. The change in velocity is proportional to $\Delta\tau$

$$\Delta v_T = \frac{\Delta\tau}{\rho c_S} \qquad \text{(III-10)}$$

In addition, the yield condition $F \leq 0$ must again be satisfied ahead of and behind the front.

The relations between the state of stress on either side of an S-front in terms of $\Delta\tau$ and of the variables s_1, s_2, β and γ can be obtained in such routine manner that only one detail used in Section IV is presented here.

It is possible for an S-front to occur between two neutral regions, i.e. regions of constant stress for both of which the yield condition, $F = 0$, is satisfied. In this special case the quantities $\mathfrak{S}$, J_1, s_1 and s_2 have no discontinuity at the front, only the direction of the principal stress changes. The values of the angles $\bar{\gamma}$, $\bar{\bar{\gamma}}$ ahead of and behind the front, respectively, are complementary

$$\bar{\bar{\gamma}} = \pi - \bar{\gamma} \qquad \text{(III-11)}$$

as shown in Fig. (5).

b) Plastic Regions.

In such regions Eqs. (II-40) apply. They are linear and homogeneous in the values s_1', s_2', etc., and may be satisfied by

$$s_1' = s_2' = J_1' = (s_1 - s_2)\,\theta' = L = 0 \qquad \text{(III-12)}$$

However, L = 0 implies $\lambda = 0$ which violates Eq. (II-9). It follows that in plastic regions the determinant of Eqs. (II-40) must vanish, giving the determinantal equation

$$(s_1 + s_2)^2 \, (b_2^2 + b_1 b_3) = 0 \tag{III-13}$$

where

$$b_1 = 2[1 + (1-2\nu)(1-2X)] \tag{III-14}$$

$$b_2 = \beta \cos 2\gamma + (1-2X)(1-2\nu) \tag{III-15}$$

$$b_3 = (1+\nu)(1-2X) - \beta^2 X \tag{III-16}$$

and

$$\beta = \frac{s_1 - s_2}{s_1 + s_2} \tag{III-17}$$

If this value is substituted it will be seen that Eq. (13) is a homogeneous quadratic expression in s_1 .

Due to the vanishing of the determinant only four of the five Eqs. (II-40) are independent. By definition L must not vanish, so that s_1' , s_2' , J_1' and θ' can always be expressed in terms of L,

$$s_1' = -\frac{(3-\beta)\, b_4 \, (s_1 + s_2)\, L}{3b_2} \tag{III-18}$$

$$s_2' = -\frac{(3+\beta)\, b_4 \, (s_1 + s_2)\, L}{3b_2} \tag{III-19}$$

$$\theta' = \frac{\sin 2\gamma \, b_3 L}{\beta(1-2X)\, b_2} \tag{III-20}$$

$$J_1' = \frac{3(1+\nu)}{(1-2\nu)} \left[b_2 - \frac{2}{3} \beta \, b_4 \right] \frac{(s_1 + s_2)\, L}{b_2} \tag{III-21}$$

Velocities and accelerations may be obtained from the relations

$$v_x' = \frac{- V \sin \varphi \, (s_1 + s_2) \, L}{2G(1-2X) \, b_2} \left[b_2\beta \sin (2\gamma - \varphi) - 2b_3 \sin \varphi \right] \qquad \text{(III-22)}$$

$$v_y' = \frac{- V \sin \varphi \, (s_1 + s_2) \, L}{2G(1-2X) \, b_2} \left[b_2\beta \cos (2\gamma - \varphi) + 2b_3 \cos \varphi \right] \qquad \text{(III-23)}$$

$$\dot{v}_x = \frac{V}{R} \sin \varphi \; v_x' \qquad \text{(III-24)}$$

$$\dot{v}_y = \frac{V}{R} \sin \varphi \; v_y' \qquad \text{(III-25)}$$

where

$$b_4 = (1+\nu) \cos 2\gamma + \beta X(1-2\nu) \qquad \text{(III-26)}$$

Since Eq. (13) must remain valid throughout a plastic region, it may be differentiated with respect to φ. This leads to an expression which contains the first derivatives of the stresses linearly, so that substitution of Eqs. (18)-(20) furnishes a linear equation for L. Its solution gives L as a function of β, γ and of the position angle φ,

$$L = \frac{3b_2(1-2X)}{4 \sin^2 \varphi} \left\{ \frac{X \sin 2\varphi \, [4(1-2\nu)(b_2 + b_3) + b_1(\beta^2 + 2 + 2\nu)] + 4b_2\beta \sin 2\gamma \sin^2 \varphi}{(3+\beta^2)(1-2X) \, b_4 \, (b_2 \cos 2\gamma - b_1\beta X) + 3b_2b_3 \sin^2 2\gamma} \right\}$$

(III-27)

The values of the derivatives s_1' , s_2' , J_1' and θ' can be obtained by substitution of Eq. (27) into Eqs. (18)-(25).

In principle Eqs. (18)-(25) permit the numerical determination of the values of stresses and velocities in the interior of a plastic region by quadratures if the values on one boundary of this region are known. The starting values must inherently satisfy the yield condition, F = 0, and the determinantal equation (13). Further, throughout the plastic region,

$$L > 0 \tag{III-28}$$

must be satisfied.

c) Discontinuities (Plastic Shock Fronts).

It is known that in transient problems one, but just one type of plastic shock front can propagate in the elastic-plastic material considered here [8]. However, such a front can exist only in locations where the normal to the front lies in the direction of one of the principal stresses, while the other two are equal and where the yield condition is satisfied. The velocity of propagation of the front is

$$\bar{c} = \sqrt{\frac{K}{\rho}} \tag{III-29}$$

where $K = \frac{2(1+\nu)}{3(1-2\nu)} G$ is the bulk modulus. The discontinuity is restricted to the particle velocity v_N normal to the front and to the first invariant J_1 . The change ΔJ_1 must have the same sign as J_1

$$\frac{\Delta J_1}{J_1} > 0 \tag{III-30}$$

The other conditions stated define the values of γ and β

$$\gamma = \frac{\pi}{2} \quad , \quad \beta = 3 \tag{III-31}$$

A discontinuity traveling in real space with the velocity $\bar{c}$, Eq. (29), can occur in the steady-state problem only in the location

$$\bar{\varphi} = \pi - \sin^{-1} \left(\frac{\bar{c}}{V}\right) \tag{III-32}$$

The corresponding value of X is

$$\bar{X} = \frac{1+\nu}{3(1-2\nu)} \tag{III-33}$$

The denominator in Eq. (27) vanishes, as expected, for these values of γ, β and X.

The possibility of the occurrence of this discontinuity (plastic shock front) must be considered when constructing the complete solutions in Section IV. It was actually found that no such shocks occur, except in the limit, $V \to \infty$. However, for large values of the parameter $\frac{p_o}{k}$ defining the surface load, the solutions come extremely close to the singular values representing a shock, so that computing difficulties occur.

d) Asymptotic Solutions near Singularities.

As stated in the previous paragraph numerical difficulties in the vicinity of $\varphi = \bar{\varphi}$ will make the procedure for integration of Eqs. (18-27) outlined in subsection b unsuitable if the values of β and γ are sufficiently close to those for a plastic front, $\beta = 3$, $\gamma = \frac{\pi}{2}$.

To establish the behavior of the solution of Eqs. (18-27) in such cases, let

$$\gamma = \frac{\pi}{2} + \eta$$
$$\beta = 3 + \Delta \qquad \text{(III-34)}$$
$$\varphi = \bar{\varphi} + \epsilon$$

where η, Δ and ϵ are small quantities. Substitution of Eqs. (34) into the determinantal equation (13), Eq. (20) for θ', and an equation for $\frac{\partial\beta}{\partial\epsilon} = \Delta'$ obtained from a combination of Eqs. (17), (18) and (19), provides three simultaneous first order differential equations for the three unknowns η, Δ and L, subject to the inequality $L > 0$. These three nonlinear differential equations are of course equivalent to the equations in subsection b and no easier to treat. However, Δ, η and ϵ being small quantities, approximate equations may be obtained by neglecting higher order terms.

Various degrees of approximation are possible depending upon the number of higher order terms of η, Δ and ϵ which are retained. Appendix A presents the resulting set of equations when the leading and the next terms in each variable are retained. In [5] the comparable set of equations is given when only leading terms of each unknown are retained. Together with the appropriate expression for $\frac{\partial J_1}{\partial \epsilon}$ the approximate equations may be used when integrating in the vicinity of $\varphi = \bar{\varphi}$.

To obtain results for high values of $\frac{p_o}{k}$ the integration of the original differential equations (18-27) was stopped when accuracy troubles developed and the integration was continued using the approximate equations derived in Appendix A. This process was successful for all numerical cases considered. The results obtained in this manner in the vicinity of the singular point approach simple asymptotic expressions derived in [5] for $\nu \neq \frac{1}{8}$. This is shown for typical cases in Figs. 15, 17 and 18 of this reference and provides a check on the results of the complicated analysis in Appendix A.

For the special case $\nu = \frac{1}{8}$ an asymptotic expression valid for $\varphi < \varphi_S = \bar{\varphi}$ is given in Appendix B. As demonstrated on a typical case in the same Appendix, it is possible to construct solutions for $\nu = \frac{1}{8}$ for high values of $\frac{p_o}{k}$ by using this asymptotic solution for $\varphi < \varphi_S$ and a numerical integration of the original differential equations (18-27) with double precision for $\varphi > \varphi_S$.

IV CONSTRUCTION OF SOLUTIONS.

In Section IV a number of partial solutions were obtained from which the solution of the complete boundary value problem is now to be constructed. Section III-b gives the differential equations for the determination of the stresses and velocities in plastic regions; Section III-a indicates that all unknowns in nondissipative regions are constants, except for discontinuities of a prescribed nature at the locations φ_S and φ_P. In addition, as discussed in Section III-c, a shock front with plastic deformation may occur at the location $\bar{\varphi}$.

In Section II, boundary conditions and additional requirements, which the solution must satisfy, were formulated and discussed. Equations (II-13 and 14) for the prescribed surface load in terms of the variables s_1, J_1, γ and φ require either

$$s_1(\pi) + \frac{J_1(\pi)}{3} = - p_o \quad , \quad \theta(\pi) = \frac{\pi}{2} \tag{IV-1a}$$

or

$$s_2(\pi) + \frac{J_1(\pi)}{3} = - p_o \quad , \quad \theta(\pi) = 0 \tag{IV-1b}$$

A further boundary condition requires that all quantities must vanish for $\varphi < \varphi_P$, Eq. (II-15). This condition, in conjunction with the fact that a plastic region or a plastic shock can exist only in locations where the yield condition is satisfied, permits the conclusion that the change in stress from vanishing values for $\varphi < \varphi_P$ to nonvanishing values must be nondissipative. However, in nondissipative regions the stresses are constant, except for discontinuities at $\varphi = \varphi_P$ or $\varphi = \varphi_S$. A solution in which plastic deformations occur at all can therefore start only in one of the two ways described below.

Case 1. Discontinuities occur at the P- and S-fronts, where the discontinuity at φ_P satisfies the inequality

$$\sigma_1 \left[\varphi_P^{(+)}\right] < \frac{\sqrt{3}(1-\nu)k}{1-2\nu} \tag{IV-2}$$

while the discontinuity $\Delta\tau$ at φ_S is of such magnitude that the yield condition

$$F\left[\varphi_S^{(+)}\right] = 0 \tag{IV-3}$$

is satisfied. The symbols $^{(+)}$ or $^{(-)}$ indicate approach from above or below, respectively.

Case 2. A discontinuity occurs at the P-front, described by Eqs. (III-7,8), so that the yield condition is satisfied for $\varphi = \varphi_P^{(+)}$, i.e.

$$\sigma_1 \left[\varphi_P^{(+)}\right] = \frac{\sqrt{3}(1-\nu)k}{1-2\nu} \tag{IV-4}$$

In Case 1, plasticity may occur only in locations $\varphi > \varphi_S$, while, in Case 2 it may occur already for $\varphi_P < \varphi < \varphi_S$.

As a next step in the search for solutions it is helpful to consider the latter as function of the nondimensional surface pressure $\frac{p_o}{k}$, while Poisson's ratio ν and the velocity V are considered constant. For sufficiently small values of $\frac{p_o}{k}$ the solution must be entirely elastic, but as $\frac{p_o}{k}$ increases plastic regions must occur and should form a gradually changing pattern. Based on [1] one can find the limiting value $\frac{p_E}{k}$, above which entirely elastic solutions are no longer possible.

The elastic solution, shown in Fig. 6, has two discontinuities at φ_P and φ_S with regions of constant stress between φ_P and φ_S and between φ_S and the loaded surface $\varphi = \pi$. Depending on the values of the parameters ν and $\frac{V}{c_P}$, the yield condition may be reached in either of the two regions resulting in different expression for $\frac{p_E}{k}$. If the region $\varphi > \varphi_S$ controls,

$$\frac{p_E}{k} = \left[\frac{3N^2}{3N^2 - 3N\cos 2\varphi_S + (1-\nu+\nu^2)\cos^2 2\varphi_S}\right]^{\frac{1}{2}} \qquad \text{(IV-5a)}$$

where

$$N = \frac{1}{2}\left[\cos 2\varphi_S + (1-2\nu)\cos 2(\varphi_S - \varphi_P)\right] \qquad \text{(IV-5b)}$$

while

$$\frac{p_E}{k} = \left[\frac{3N^2}{(1-2\nu)^2 \cos^2 2\varphi_S}\right]^{\frac{1}{2}} \qquad \text{(IV-6)}$$

if the region $\varphi < \varphi_S$ controls. The decision which region controls can be made by comparing the values given by Eqs. (5a) and (6), the smaller one controlling. Designating as Range I the combination of values ν and $\frac{V}{c_P}$ where Eq. (5a) controls, one finds that in this range

$$\left(\frac{V}{c_P}\right)^2 > \frac{(1-2\nu)^2}{(1-\nu)(1-3\nu)} \qquad \text{(IV-7)}$$

The remainder of the range will be designated as Range II. Both ranges are shown in Fig. 7. The limiting values $\frac{p_E}{k}$ are shown in Fig. 8 for several values of ν as function of $\frac{V}{c_P}$.

If the surface load exceeds the value $\frac{p_E}{k}$ by a sufficiently small amount the elastic-plastic solution should differ only slightly from the elastic one, which can be used to predict the character of the solution. Because the situations differ, the Ranges I and II must be discussed separately.

a) <u>Solutions in Range I.</u>

In this case $\frac{p_E}{k}$ is given by Eq. (5a) and the yield stress in the elastic solution is reached only in the region $\varphi > \varphi_S$. The discontinuity σ_1 satisfies then the inequality (2) and continuity requires that this inequality will still apply for a range of values $\frac{p_L}{k} > \frac{p_o}{k} > \frac{p_E}{k}$, where p_L is a limiting value, not yet known. In this range the start of the solutions will be according to Case 1 and plasticity can therefore occur only in the region $\varphi > \varphi_S$.

Using an indirect approach, the determination of plastic solutions for values $\frac{p_o}{k} < \frac{p_L}{k}$ begins with the selection of a pair of starting values σ_1 and $\Delta\tau$ near those for the limiting elastic case. Experience and continuity considerations indicate that this value σ_1 should be larger than σ_1 corresponding to p_E given by Eq. (5a). A plastic region can start only at a point φ_1 which is a root of Eq. (III-13).

Inequalities derived in [5] indicate that, for a given state of stress, this equation has only two roots in the meaningful range of φ, namely $\frac{\pi}{2} < \varphi < \pi$, subject to the following bounds:

$$\varphi_P \leq \varphi_A \leq \varphi_S \quad , \quad \varphi_S \leq \varphi_B < \pi \tag{IV-8}$$

Starting integration at $\varphi_1 = \varphi_B$ the solution in the interior of the plastic region is determined from Eqs. (III-18 to 27). The plastic region can be extended as long as Eq. (III-27) gives values $L > 0$, but the plastic region may be terminated at will at any earlier location φ_2. The solution for $\varphi > \varphi_2$ is then nondissipative, i.e. all quantities are constant. If, therefore, in the process of forward integration a value ϑ is encountered which satisfies Eqs. (1), the plastic region is terminated and a solution for one value of the surface pressure $\frac{p_o}{k}$ has been obtained. Repeating this process with gradually increasing starting values σ_1 the whole spectrum of values satisfying the inequality (2) can be explored. Solutions, if any, obtained in this manner will have the configuration shown in Fig. 9, i.e. discontinuities at φ_P and φ_S and a plastic region $\pi > \varphi_2 > \varphi_1 > \varphi_S$. There will be an elastic region of constant stress from φ_P to φ_S and two neutral regions to either side of the plastic one.

One can proceed in a similar manner when the solution begins at $\varphi = \varphi_P$ as indicated in Case 2. Starting with a value σ_1 according to Eq. (3), the yield condition is satisfied for any value $\varphi > \varphi_P$, so that the determinantal equation (III-13) according to Eqs. (8) now has two roots, $\varphi_1 \lessgtr \varphi_S$, at which plastic regions may start. Both roots must be explored. If the larger one, $\varphi > \varphi_S$, leads to a solution, it has a configuration as shown in Fig. 10a. Starting, alternatively, with the smaller root, $\varphi_P < \varphi_1 = \varphi_A < \varphi_S$, several possibilities are to be investigated. The integration may be continued as long as $L \geq 0$ to see if a value $\theta = 0$ or $\frac{\pi}{2}$ can be reached. The configuration of such a solution, if any, is shown in Fig. 10b. Alternatively, the plastic region can be terminated at will at a point $\varphi_2 < \varphi_S$ where $\theta \neq 0, \frac{\pi}{2}$. The plastic region will then be followed by a neutral one for values $\varphi > \varphi_2$. The inequalities (8) on the roots of Eq. (III-13) indicate that there is just one more root $\varphi_3 > \varphi_S$, when a second plastic region can begin. Starting integration at this point may lead to a terminal location φ_4, where $\theta = 0$ or $\frac{\pi}{2}$. The configuration of such a solution, if any, Fig. 10c, contains a P-front and two plastic regions, separated by three neutral regions. There are, however, further possibilities. The neutral region $\varphi > \varphi_2$ which follows the first plastic one may be terminated at φ_S by an elastic change in shear, $\Delta\tau$, which is restricted in sign and intensity by the yield condition. If $\Delta\tau$ is such that $F\left[\varphi_S^{(+)}\right] < 0$, the region $\varphi > \varphi_S$ becomes elastic. This might permit values $\theta = 0, \frac{\pi}{2}$ at the surface, the corresponding solution having the configuration of Fig. 11. Finally, the important case must be considered where the value of $\Delta\tau$ is such that $F\left[\varphi_S^{(+)}\right]$ vanishes again, a situation discussed in Section III-a-2. In the latter case there is again a neutral region for $\varphi > \varphi_S$, which can be followed by a plastic region because Eq. (III-13) has a root $\varphi_3 > \varphi_S$ giving a starting location. The configuration of a solution obtained in this manner is shown in Fig. 12.

b) Solutions in Range II.

In this range $\frac{p_E}{k}$ is given by Eq. (6) so that in the limiting case $\frac{p_o}{k} = \frac{p_E}{k}$ yield is just reached in the region $\varphi_P < \varphi < \varphi_S$. The discontinuity σ_1 at the P-front must therefore satisfy Eq. (4), which will also hold for neighboring elastic-plastic solutions where $\frac{p_o}{k}$ exceeds $\frac{p_E}{k}$ slightly. These solutions will therefore start at $\varphi = \varphi_P$ according to Case 2. In the limiting solution for $\frac{p_o}{k} = \frac{p_E}{k}$ the region $\varphi > \varphi_S$ is below yield and continuity requires this to hold in neighboring elastic-plastic solutions, so that the plastic region must lie in the range $\varphi_P < \varphi_1 < \varphi_2 < \varphi_S$. The construction of solutions begins exactly as for $\frac{p_o}{k} > \frac{p_L}{k}$. For each terminal point φ_2 the strength of the discontinuity $\Delta\tau$ at the shear front is determined by the requirement that $\theta = 0$ or $\frac{\pi}{2}$ subject to the limitation $F\left[\varphi_S^{(+)}\right] \leq 0$. When the required value $\Delta\tau$ violates this condition a second plastic region for $\varphi > \varphi_S$ is needed, i.e. the configurations shown in Figs. 10c and 12 are to be investigated.

c) Alternative Solutions and Considerations of Uniqueness and Existence.

In the absence of a uniqueness theorem it is vital to demonstrate that configurations other than those in Figs. 9 to 12 can not lead to solutions. According to [5], Eq. (III-13) has for a given state of stress one root φ, no more no less, in each of the intervals $\varphi_P < \varphi < \varphi_S$ and $\varphi_S < \varphi < \pi$. If a plastic region ends at a location $\bar{\varphi}_1$ in one of these intervals, the state of stress in the remainder of the interval for $\varphi > \varphi_1$ is necessarily neutral and uniform and equal to the one at the terminal point φ_1 of the plastic region. φ_1 is therefore the only solution of Eq. (III-13) for this state of stress in the particular interval and no more than one plastic zone can therefore occur in any interval.

In Section III-3 the possibility of discontinuous plastic shock fronts has been indicated and their occurrence must be considered. It has been shown in [5] that for finite values of V no plastic shock front can occur and, that there can be no more than one plastic region in each of the intervals $\varphi_P < \varphi < \varphi_S$, $\varphi_S < \varphi < \pi$. While discontinuous plastic shock fronts can not occur, values of γ and β, where the conditions (III-31) are nearly satisfied, are encountered. The asymptotic behavior of the solution near $\varphi = \bar{\varphi}$ in such cases was studied in Section III-d and details are given in Section 5 of [5]. Therefore, combined with elastic discontinuities at the P- and S-fronts, only the limited number of configurations shown in Figs. 9 to 12 are possible.

The numerical analysis by digital computer was set up to investigate all possible alternatives, i.e. the configuration according to Fig. 9 if the starting value σ_1 satisfies Eq. (2) and any of the alternatives shown in Figs. 10 to 12 if σ_1 satisfies Eq. (3). While none of the configurations shown in Figs. 10a-c ever furnished a solution, no general proof permitting elimination of these cases is available.

In Range I solutions which start according to Case 1, have the configuration of Fig. 9. For fixed values of ν and V, these solutions form a family which depends on one parameter, the selected starting value $\sigma_1 > \sigma_{1E}$. It was found that the surface load $\frac{p_o}{k}$ increases monotonically with σ_1 until the limit, Eq. (4) for σ_1 is reached, which leads to a limiting value of the surface load $\frac{p_L}{k}$. However, no analytical proof of the monotonic increase of $\frac{p_o}{k}$ is available.

The solutions found for Range I, which start according to Case 2, had always the configuration shown in Fig. 12. These solutions also depend on one parameter, viz. the stopping point φ_2 of the plastic region between φ_P and φ_S . If φ_2 is selected only slightly larger than φ_1 , the solution must obviously be very close to the limiting one for Case 1, so that in such a case $\frac{p_o}{k} \gtrsim \frac{p_L}{k}$ and there is a smooth transition from the configuration according to Fig. 9 to that of Fig. 12. The numerical analysis indicated that the surface load $\frac{p_o}{k}$ increases monotonically with φ_2 . As φ_2 approaches a limiting value the surface load goes to the limit $\frac{p_o}{k} \to \infty$, for reasons explained in Section III-d.

In Range II, only solutions which start according to Case 2 were found, their configurations being as shown in Figs. 11 and 12. Figure 11 applied for values $\frac{p_E}{k} < \frac{p_o}{k} \leq \frac{p_L}{k}$ where $\frac{p_L}{k}$ is a bound. The corresponding family of solutions depends on the stopping point φ_2 of the plastic region. The bound $\frac{p_L}{k}$ is reached when the elastic region for $\varphi > \varphi_S$ becomes neutral. For larger values of $\frac{p_o}{k}$ Fig. 12 applies and all statements made in Range 1 for this case apply.

In Range I as well as in Range II, combination of all solutions obtained numerically furnished one, and only one solution for each value of $\frac{p_o}{k} > \frac{p_E}{k}$. However, no general proof is available that this must be so. Existence and uniqueness of the solutions obtained must therefore be demonstrated for each combination of values ν and $\frac{V}{c_P}$ by actual computation of the families of solution according to the configurations shown in Figs. 9 to 12.

V RESULTS AND CONCLUSIONS.

a) Numerical Results for the Stresses.

For the numerical integration of the simultaneous differential equations (III-18 to 21) in plastic regions, a Runge-Kutta forward integration scheme of Fourth order, [7], was used. Computations were programmed in FORTRAN for an IBM 7090 digital computer. Results for the stresses are given in Tables 1 to 16, (to be used in conjunction with Fig. 13), for all combinations of the parameters $\nu = 0, 0.125, 0.25, 0.35$ and $\frac{V}{c_P} = 1.25, 1.5, 2.0, 4.0$, for five different values of $\frac{p_o}{k}$. The values $\frac{p_o}{k}$ have been selected such that there are a sufficient number of results to permit interpolation in each of the configurations applicable for each of the combinations of ν and $\frac{V}{c_P}$.

In general there are three distinctive values of $\frac{p_o}{k}$ for each case, which are called $\frac{p_E}{k}$, $\frac{p_L}{k}$ and $\frac{p_H}{k}$. The value $\frac{p_E}{k}$ is the one up to which the solution is entirely elastic. The distinctive value, $\frac{p_L}{k}$, defines the range $\frac{p_E}{k} < \frac{p_o}{k} \leq \frac{p_L}{k}$ where a configuration with one plastic region applies according to either Fig. 9 or Fig. 11. For $\frac{p_o}{k} > \frac{p_L}{k}$ the applicable configuration, Fig. 12, contains two plastic regions.

As mentioned in Section III computational difficulties arise in the case $\frac{p_o}{k} > \frac{p_L}{k}$ when $\frac{p_o}{k}$ becomes larger than about 5. In the integration the value of $\frac{p_o}{k}$ obtained depends on the selected end point φ_2 of the lower plastic region. The details of the difficulties depend on the value of Poisson's ratio, $\nu \gtrless \frac{1}{8}$. For $\nu > \frac{1}{8}$, φ_2 remains smaller than $\bar{\varphi}$, but the computation becomes sensitive when φ_2 approaches $\bar{\varphi}$. For $\nu < \frac{1}{8}$ the point $\bar{\varphi}$ is situated in the interior of the upper plastic region, but the computation in the vicinity of $\bar{\varphi}$ becomes very sensitive to small changes in φ_2. For $\nu = \frac{1}{8}$, when $\bar{\varphi} = \varphi_8$, both plastic regions approach $\bar{\varphi}$, and the computation again becomes very sensitive to small changes in φ_2.

In either of these cases the sensitivity is due to a very rapid change, in the proximity of $\bar{\varphi}$, in the quantity J_1 . This change is extremely rapid, nearly a shock front, while other quantities near $\bar{\varphi}$ approach limiting values smoothly.

For values $\frac{p_o}{k}$ where computational difficulties arise, the approximate differential equations (A-5-7) derived in Appendix A must be used. One result, designated by $\frac{p_H}{k}$, requiring this type of analysis is given in the tables for each combination of ν and $\frac{V}{c_P}$. The stresses σ_1 , σ_2 and J_1 for any higher value of the surface pressure, say $\frac{p_o}{k} = \frac{(p_H + \delta)}{k}$, are equal to $\sigma_1^H + \delta$, $\sigma_2^H + \delta$, $J_1^H + \delta$, respectively, at locations $\varphi > \bar{\varphi}$, where the superscript H indicates the respective values for p_H given in the tables. The stresses for location $\varphi < \bar{\varphi}$, and the quantities β and θ everywhere, are approximately equal to those for p_H given in the tables.

The numerical results for the stresses are recorded in Tables 1-16. The notation "Not Applicable" in the tables which may appear in Regions B or C, indicates that the respective region does not occur at all for this value of p_o , i.e. region A or D, respectively, or both extend as far as the S-front. For values $\frac{p_o}{k} < \frac{p_H}{k}$ results may be obtained by interpolation (sometimes nonlinear), while for $\frac{p_o}{k} > \frac{p_H}{k}$ the procedure stated in the previous paragraph is to be used. At points of transition in the configuration, i.e. for the loads p_E and p_L , it is seen that the upper and lower limits of one of the plastic regions are equal, $\varphi_1 = \varphi_2$, or $\varphi_3 = \varphi_4$ indicating that there is actually no plastic region. These values are given in the tables to permit interpolation for pressures exceeding the respective value p_E or p_L .

b) Simplified Determination of Velocities and Accelerations.

The basic relations in Section II permit the numerical determination of stresses and velocities or accelerations. The integration for the stresses must be actually carried out to satisfy the boundary condition on the surface. The parallel integrations to find velocities and accelerations may be avoided, by using the following relations, some of which are exact, while others are only good approximations.

At a front of discontinuities, i.e. a P- or S-front, the accelerations are of course infinite, but the changes in velocity are given - exactly - in terms of the respective stress discontinuities,

$$\text{at } \varphi = \varphi_P \qquad \left| \Delta\dot{u}_N \right| = \frac{\Delta\sigma_1}{\rho \ V \sin \varphi_P} \tag{V-1}$$

$$\text{at } \varphi = \varphi_S \qquad \left| \Delta\dot{u}_T \right| = \frac{\Delta\tau}{\rho \ V \sin \varphi_S} \tag{V-2}$$

where the subscripts N, T indicate normal and tangential directions, respectively. The value of $\Delta\sigma_1$ at a P-front can be taken directly from the numerical computations for the stresses. The value $\Delta\tau$ can easily be computed and is given in the tables listing numerical results.

In con''nuous elastic regions velocities do not change, while accelerations vanish. Inelastic regions being reasonably narrow, one may disregard tangential accelerations and changes in velocity, while the normal acceleration may be assumed to be uniform in the region, giving a linear change in velocity. The total change in velocity, Δu_N, in an inelastic region of extent $\Delta\varphi$, may be found from the change, $\Delta\sigma$, in the principal stress σ_1 at both ends of the region

$$\left| \Delta\dot{u}_N \right| \simeq \left| \frac{\Delta\sigma}{\rho \ V \sin \varphi_m} \right| \tag{V-3}$$

where φ_m is an angle defining the location of the plastic region, say the mean of the values φ_j at the end points. The acceleration is

$$\left| \ddot{u}_N \right| \simeq \left| \frac{\Delta\dot{u}_N}{\Delta\varphi} \right| \tag{V-4}$$

Except for values of $\frac{V}{c_p}$ very close to unity, the plastic regions are narrow, so that the above approximations are quite satisfactory when determining shock factors.

c) Conclusions.

A method has been presented and numerical results have been tabulated for the problem of a half-space of a von Mises material subject to a step load progressing with superseismic velocity, $V > c_p$.

With regard to application of the results to protective construction, it must be emphasized that the effects of a step load and those of a decaying pressure differ necessarily. The results for the step load may be used approximately for a decaying load only for points in the vicinity of the front of the surface load and soon after passing of this front. The solution for the step load approximates the solution for the decaying load reasonably within a distance D from the front, Fig. 14. This distance is the distance in which the peak pressure decays by about 10-20%.

It should be noted that the present report emphasizes numerical results. More theoretical and academic matters, e.g. concerning uniqueness, are treated in [5].

APPENDIX A - Approximate Forms of Equations (III-18-21 and 27).

Using the relations

$$\gamma = \frac{\pi}{2} + \eta$$
$$\beta = 3 + \Delta \qquad \text{(A-1)}$$
$$\varphi = \bar{\varphi} + \epsilon$$

A set of approximate relations will be derived to replace Eqs. (III-18-21 and 27) in the vicinity of $\varphi \to \bar{\varphi}$ where the numerical integration of the above equations encounters difficulties. η, Δ and ϵ are to be considered small, but, to obtain sufficient range of validity of this approximation the first two significant terms in each variable are retained.

Replacing $\epsilon = \bar{\varphi} - \varphi$ by the more convenient variable

$$\xi = \bar{X} - X \qquad \text{(A-2)}$$

where X, $\bar{X}$ are defined by Eqs. (II-34), (III-33), respectively. To the order of the approximation used one finds

$$\xi \simeq -\frac{2(1+\nu)}{3(1-2\nu)}\left[\sqrt{\frac{3(1-\nu)\,V^2}{(1+\nu)\,c_p^2} - 1} - \frac{3V^2(1-\nu)}{2c_p^2\,(1+\nu)}\,\epsilon\right]\epsilon \qquad \text{(A-3)}$$

Similarly, approximate expressions for $\cos 2\gamma$, $\sin 2\gamma$ are

$$\cos 2\gamma = -1 + 2\eta^2$$
$$\sin 2\gamma = -2\eta \qquad \text{(A-4)}$$

Introducing Eqs. (A-1) to (A-4) into the determinantal equation (III-13) and the differential equation (III-20) for θ', retaining only appropriate terms yields

$$a_1\Delta^2 + a_2\eta^2 = a_3\xi \qquad \text{(A-5)}$$

$$\frac{d\eta}{de} = 1 + \frac{a_4}{a_5} \eta L \qquad \text{(A-6)}$$

Combination of Eqs. (III-17, 18 and 19) permits formulation of a relation for $\frac{d\Delta}{de} = \beta'$,

$$\frac{d\Delta}{de} = \frac{a_6 a_7}{a_8} L \qquad \text{(A-7)}$$

where

$$a_1 = \frac{1-8\nu}{9} - 4\eta^2 - \frac{4}{3}(1-5\nu)\xi + 4(1-2\nu)\xi^2 \qquad \text{(A-8)}$$

$$a_2 = -32(1+\nu) - 24(1-2\nu)\xi - \frac{4}{3}(17+8\nu)\Delta - 8(1-2\nu)\xi\Delta \qquad \text{(A-9)}$$

$$a_3 = 8(1-8\nu) - 48(1-2\nu)\xi + 4(1-8\nu)\Delta - 24(1-2\nu)\xi\Delta \qquad \text{(A-10)}$$

$$a_4 = 4(1+\nu) + \frac{3}{2}\Delta + 3(1-2\nu)\xi - 9\eta^2 - 3\eta^2\Delta \qquad \text{(A-11)}$$

$$a_5 = 2(1-8\nu) - 3(5-16\nu)\xi + \frac{2(1-8\nu)}{3}\Delta - (5-16\nu)\xi\Delta + $$
$$+ 18\xi^2(1-2\nu) + 6(1-2\nu)\xi^2\Delta \qquad \text{(A-12)}$$

$$a_6 = 12 + 6\Delta + \Delta^2 \qquad \text{(A-13)}$$

$$a_7 = 3(1-2\nu)\xi + \frac{(1+\nu)}{3}\Delta + 2(1+\nu)\eta^2 + (1-2\nu)\xi\Delta \qquad \text{(A-14)}$$

$$a_8 = -4(1+\nu) - \frac{3}{2}\Delta + 9\eta^2 - 3(1-2\nu)\xi + 3\Delta\eta^2 \qquad \text{(A-15)}$$

Equations (A-5 to 7) govern the solution in terms of the three unknowns η, Δ and $L > 0$.

The derivative of J_1 becomes

$$\frac{dJ_1}{de} = \frac{3(1+\nu)}{1-2\nu}\left[1 - \frac{2}{3}\frac{a_9}{a_{10}}\right](s_1 + s_2) L \qquad \text{(A-16)}$$

where

$$a_9 = 9(1-2\nu)\,\xi + (1+\nu)\,\Delta + 6(1+\nu)\,\eta^2 + \frac{(1+\nu)}{3}\,\Delta^2 + 6(1-2\nu)\,\Delta\xi + (1-2\nu)\,\Delta^2\xi + 2(1+\nu)\,\Delta\eta^2 \tag{A-17}$$

$$a_{10} = -\frac{8(1+\nu)}{3} - \Delta - 2(1-2\nu)\,\xi + 2\Delta\eta^2 \tag{A-18}$$

The quantities s_1 and s_2 can be found from the yield condition and from the relation $\beta = 3 + \Delta$, once Δ is found.

It may be shown easily that Eqs. (A-5,6) reduce to their counterparts presented in [5] when ξ, Δ and η become small enough to permit retention of leading terms only. Equation (A-7) also reduces to its counterpart in [5] providing the relationship between ξ, η and Δ given by the simplified determinantal equation {Eq. (3-35) of [5]} is employed.

APPENDIX B - Solutions for $\nu = \frac{1}{8}$ near $\varphi = \varphi_S \equiv \bar{\varphi}$.

For $\nu = \frac{1}{8}$ the asymptotic solutions valid for $\varphi < \varphi_S \equiv \bar{\varphi}$ can be obtained from a simplification of the differential equations (A-5, 6 and 7) where, after substitution of $\nu = \frac{1}{8}$, only the leading terms have been retained:

$$36\,\eta^2 + \tfrac{1}{2}\,\xi\Delta^2 = 36\xi^2 \qquad \text{(B-1)}$$

$$\Delta' = -\,(6\xi + \Delta)\,L \qquad \text{(B-2)}$$

$$\eta' = 1 - \frac{\eta}{2\xi}\,L \qquad \text{(B-3)}$$

In a manner similar to the one used in [5], an asymptotic solution for $\epsilon < 0$ of these equations is

$$\Delta = -\,6\sqrt{3}\,\xi^{\frac{3}{2}} \qquad \text{(B-4)}$$

$$\eta \doteq 0 \qquad \text{(B-5)}$$

Using this asymptotic solution for $\varphi < \varphi_S \equiv \bar{\varphi}$ and a double accuracy integration of the original differential equations (III-18 to 27) for $\varphi > \varphi_S \equiv \bar{\varphi}$, solutions could be constructed successfully. As a typical case, Figs. 15 and 16 show the values of Δ and η as functions of ϵ for $V = 1.25\ c_P$. There are only two extremely narrow neutral regions between the two plastic regions and the S-front. Further, the discontinuity in shear at the S-front becomes extremely small, approximately zero.

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TABLE 1 (For definition of regions, see Fig. 13)

$\nu = 0.0$ $V = 1.25\ c_p$ $\varphi_p = 126.87^\circ$ $\varphi_S = 145.55^\circ$ $\bar{\varphi} = 152.49^\circ$

		p_E $p_o = 1.41\ k$	$p_E < p_o < p_L$ $p_o = 2.17\ k$	p_L $p_o = 2.40\ k$	$p_L < p_o < p_H$ $p_o = 3.20\ k$	p_H $p_o = 8.02\ k$
Region A	σ_1/k	- 0.876	- 1.58	- 1.73	- 1.73	- 1.73
	σ_2/k	0	0	0	0	0
	J_1/k	- 0.876	- 1.58	- 1.73	- 1.73	- 1.73
	β	3.00	3.00	3.00	3.00	3.00
	θ	36.87°	36.87°	36.87°	36.87°	36.87°
	φ_1	Not Applicable	Not Applicable	140.29°	140.29°	140.29°
	φ_2	"	"	140.29°	143.51°	144.08°
Region B	σ_1/k	"	"	Not Applicable	- 1.86	- 1.89
	σ_2/k	"	"	"	- 0.170	- 0.211
	J_1/k	"	"	"	- 2.11	- 2.22
	β	"	"	"	2.72	2.69
	θ	"	"	"	46.29°	48.6°
	$\Delta\tau/k$	- 1.17	- 1.11	- 1.05	- 0.536	- 0.404
Region C	σ_1/k	Not Applicable	- 1.68	- 1.73	- 1.86	- 1.89
	σ_2/k	"	0.096	0	- 0.170	- 0.211
	J_1/k	"	- 1.58	- 1.73	- 2.11	- 2.22
	β	"	3.37	3.00	2.72	2.69
	θ	"	78.00°	74.23°	64.81°	62.5°
	φ_3	161.39°	156.8°	155.42°	152.67°	152.42°
	φ_4	161.39°	161.39°	161.39°	161.40°	161.40°
Region D	σ_1/k	- 1.41	- 2.17	- 2.40	- 3.20	- 8.02
	σ_2/k	0.529	- 0.333	- 0.569	- 1.36	- 6.24
	J_1/k	- 0.876	- 3.08	- 3.77	- 6.16	- 20.7
	β	6.62	4.06	3.98	4.01	4.04
	θ	90°	90°	90°	90°	90°

TABLE 2 (For definition of regions, see Fig. 13)

$\nu = 0.0$ $V = 1.5$ c_p $\varphi_P = 138.19^\circ$ $\varphi_S = 151.87^\circ$ $\bar{\varphi} = 157.36^\circ$

		p_E $p_o = 1.53k$	$p_E < p_o < p_L$ $p_o = 1.85k$	p_L $p_o = 2.38k$	$p_L < p_o < p_H$ $p_o = 2.71k$	p_H $p_o = 6.57k$
Region A	σ_1/k	- 1.18	- 1.44	- 1.73	- 1.73	- 1.73
	σ_2/k	0	0	0	0	0
	J_1/k	- 1.18	- 1.44	- 1.73	- 1.73	- 1.73
	β	3.00	3.00	3.00	3.00	3.00
	θ	48.19°	48.19°	48.19°	48.19°	48.19°
	φ_1	Not Applicable	Not Applicable	148.90°	148.90°	148.90°
	φ_2	"	"	148.90°	150.02°	151.09°
Region B	σ_1/k	"	"	Not Applicable	- 1.78	- 1.83
	σ_2/k	"	"	"	- 0.0689	- 0.124
	J_1/k	"	"	"	- 1.89	- 2.02
	β	"	"	"	2.88	2.81
	θ	"	"	"	52.15°	56.38°
	$\Delta\tau/k$	- 1.05	- 0.978	- 0.796	- 0.569	- 0.327
Region C	σ_1/k	Not Applicable	- 1.63	- 1.73	- 1.78	- 1.83
	σ_2/k	"	0.190	0	- 0.0689	- 0.124
	J_1/k	"	- 1.44	- 1.73	- 1.89	- 2.02
	β	"	3.79	3.00	2.88	2.81
	θ	"	84.57°	75.56°	71.60°	67.37°
	φ_3	163.48°	161.72°	158.88°	157.89°	157.34°
	φ_4	163.48°	163.53°	163.47°	163.48°	163.48°
Region D	σ_1/k	- 1.53	- 1.85	- 2.38	- 2.71	- 6.57
	σ_2/k	0.352	- 0.135	- 0.571	- 0.893	- 4.79
	J_1/k	- 1.18	- 2.11	- 3.70	- 4.67	- 16.32
	β	4.80	4.00	3.72	3.72	3.75
	θ	90°	90°	90°	90°	90°

TABLE 3 (For definition of regions, see Fig. 13)

$\nu = 0.0$ $V = 2.00\ c_P$ $\varphi_P = 150.00^\circ$ $\varphi_S = 159.30^\circ$ $\bar{\varphi} = 163.22^\circ$

		p_E $p_o = 1.63k$	$p_E < p_o < p_L$ $p_o = 1.95k$	p_L $p_o = 2.31k$	$p_L < p_o < p_H$ $p_o \approx 2.72k$	p_H $p_o = 6.61k$
Region A	σ_1/k	- 1.44	- 1.62	- 1.73	- 1.73	- 1.73
	σ_2/k	0	0	0	0	0
	J_1/k	- 1.44	- 1.62	- 1.73	- 1.73	- 1.73
	β	3.00	3.00	3.00	3.00	3.00
	θ	60.00°	60.00°	60.00°	60.00°	60.00°
	φ_1	Not Applicable	Not Applicable	157.95°	157.95°	157.95°
	φ_2	"	"	157.95°	158.66°	158.95°
Region B	σ_1/k	"	"	Not Applicable	- 1.77	- 1.78
	σ_2/k	"	"	"	- 0.0495	- 0.0671
	J_1/k	"	"	"	- 1.85	- 1.89
	β	"	"	"	2.92	2.90
	θ	"	"	"	63.57°	65.40°
	$\Delta\tau/k$	- 0.831	- 0.696	- 0.552	- 0.342	- 0.237
Region C	σ_1/k	Not Applicable	- 1.69	- 1.73	- 1.77	- 1.78
	σ_2/k	"	0.0741	0	- 0.0495	- 0.0671
	J_1/k	"	- 1.62	- 1.73	- 1.85	- 1.89
	β	"	3.27	3.00	2.92	2.90
	θ	"	84.16°	78.59°	75.02°	73.26°
	φ_3	166.54°	165.08°	163.84°	163.31°	163.22°
	φ_4	166.54°	166.56°	166.54°	166.54°	166.54°
Region D	σ_1/k	- 1.63	- 1.95	- 2.31	- 2.72	- 6.61
	σ_2/k	0.189	- 0.161	- 0.523	- 0.932	- 4.84
	J_1/k	- 1.44	- 2.40	- 3.47	- 4.69	- 16.39
	β	3.79	3.50	3.43	3.43	3.44
	θ	90°	90°	90°	90°	90°

TABLE 4 (For definition of regions, see Fig. 13)

$\nu = 0.0$ $V = 4.00\ c_p$ $\varphi_p = 165.52^\circ$ $\varphi_S = 169.82^\circ$ $\bar{\varphi} = 171.70^\circ$

		p_E $p_o = 1.71k$	$p_E < p_o < p_L$ $p_o = 1.91k$	p_L $p_o = 2.17k$	$p_L < p_o < p_H$ $p_o = 2.60k$	p_H $p_o = 7.12k$
Region A	σ_1/k	- 1.66	- 1.71	- 1.73	- 1.73	- 1.73
	σ_2/k	0	0	0	0	0
	J_1/k	- 1.66	- 1.71	- 1.73	- 1.73	- 1.73
	β	3.00	3.00	3.00	3.00	3.00
	θ	75.52°	75.52°	75.52°	75.52°	75.52°
	φ_1	Not Applicable	Not Applicable	169.61°	169.61°	169.61°
	φ_2	"	"	169.61°	169.74°	169.77°
Region B	σ_1/k	"	"	Not Applicable	- 1.74	- 1.75
	σ_2/k	"	"	"	- 0.0145	- 0.018
	J_1/k	"	"	"	- 1.77	- 1.77
	β	"	"	"	2.98	2.97
	θ	"	"	"	77.34°	77.89°
	$\Delta\tau/k$	- 0.429	- 0.341	- 0.259	- 0.149	- 0.116
Region C	σ_1/k	Not Applicable	- 1.72	- 1.73	- 1.74	- 1.75
	σ_2/k	"	0.0171	0	- 0.0145	- 0.18
	J_1/k	"	- 1.71	- 1.73	- 1.77	- 1.77
	β	"	3.06	3.00	2.98	2.97
	θ	"	86.93°	84.11°	82.29°	81.74°
	φ_3	172.38°	172.02°	171.78°	171.71°	171.70°
	φ_4	172.38°	172.38°	172.38°	172.38°	172.38°
Region D	σ_1/k	- 1.71	- 1.91	- 2.17	- 2.60	- 7.12
	σ_2/k	0.0455	- 0.157	- 0.425	- 0.856	- 5.37
	J_1/k	- 1.66	- 2.26	- 3.05	- 4.35	- 17.9
	β	3.16	3.13	3.11	3.11	3.12
	θ	90°	90°	90°	90°	90°

TABLE 5 (For definition of regions, see Fig. 13)

$\nu = 0.125$ $V = 1.25$ c_P $\varphi_P = 126.87^\circ$ $\varphi_S = 148.42^\circ$ $\bar{\varphi} = 148.42^\circ$

		p_E $p_o = 1.72k$	$p_E < p_o < p_L$ $p_o = 2.09k$	p_L $p_o = 2.36k$	$p_L < p_o < p_H$ $p_o = 3.65k$	p_H $p_o = 6.14k$
Region A	σ_1/k	- 1.36	- 1.75	- 2.02	- 2.02	- 2.02
	σ_2/k	- 0.194	- 0.2[illegible]1	- 0.289	- 0.289	- 0.289
	J_1/k	- 1.75	- 2.26	- 2.60	- 2.60	- 2.60
	β	3.00	3.00	3.00	3.00	3.00
	θ	36.87°	36.87°	36.87°	36.87°	36.87°
	φ_1	Not Applicable	Not Applicable	140.38°	140.38°	140.38°
	φ_2	"	"	140.38°	148.32°	148.42°
Region B	σ_1/k	"	"	Not Applicable	- 2.64	- 5.15
	σ_2/k	"	"	"	- 0.958	- 3.47
	J_1/k	"	"	"	- 4.47	- 12.08
	β	"	"	"	2.72	2.99
	θ	"	"	"	58.19°	58.42°
	$\Delta\tau/k$	- 1.24	- 1.23	- 1.16	- 0.0135	0
Region C	σ_1/k	Not Applicable	- 1.90	- 2.02	- 2.64	- 5.15
	σ_2/k	"	- 0.102	- 0.289	- 0.958	- 3.47
	J_1/k	"	- 2.26	- 2.60	- 4.47	- 12.08
	β	"	3.59	3.00	2.72	2.99
	θ	"	84.65°	79.97°	58.64°	58.42°
	φ_3	160.90°	158.62°	156.59°	148.49°	148.42°
	φ_4	160.90°	160.83°	160.80°	160.85°	160.85°
Region D	σ_1/k	- 1.72	- 2.09	- 2.36	- 3.65	- 6.14
	σ_2/k	0.165	- 0.248	- 0.541	- 1.80	- 4.34
	J_1/k	- 1.75	- 2.83	- 3.65	- 7.51	- 15.1
	β	4.85	4.06	3.85	4.14	4.25
	θ	90°	90°	90°	90°	90°

TABLE 6 (For definition of regions, see Fig. 13)

$\nu = 0.125$ $V = 1.50\ c_p$ $\varphi_p = 138.19^\circ$ $\varphi_S = 154.12^\circ$ $\bar{\varphi} = 154.12^\circ$

		p_E $p_o = 1.82k$	$p_E < p_o < p_L$ $p_o = 2.04k$	p_L $p_o = 2.35k$	$p_L < p_o < p_H$ $p_o = 3.26k$	p_H $p_o = 5.95k$
Region A	σ_1/k	- 1.57	- 1.77	- 2.02	- 2.02	- 2.02
	σ_2/k	- 0.224	- 0.253	- 0.289	- 0.289	- 0.289
	J_1/k	- 2.02	- 2.28	- 2.60	- 2.60	- 2.60
	β	3.00	3.00	3.00	3.00	3.00
	θ	48.19°	48.19°	48.19°	48.19°	48.19°
	φ_1	Not Applicable	Not Applicable	149.05°	149.05°	149.05°
	φ_2	"	"	149.05°	153.81°	154.12°
Region B	σ_1/k	"	"	Not Applicable	- 2.46	- 5.14
	σ_2/k	"	"	"	- 0.759	- 3.44
	J_1/k	"	"	"	- 3.92	- 12.01
	β	"	"	"	2.80	2.99
	θ	"	"	"	63.16°	64.12°
	$\Delta\tau/k$	- 1.08	- 1.03	- 0.914	- 0.0569	0
Region C	σ_1/k	Not Applicable	- 1.91	- 2.02	- 2.46	- 5.14
	σ_2/k	"	- 0.113	- 0.289	- 0.759	- 3.44
	J_1/k	"	- 2.28	- 2.60	- 3.92	- 12.01
	β	"	3.55	3.00	2.80	2.99
	θ	"	86.22°	80.06°	65.08°	64.12°
	φ_3	162.89°	161.58°	159.36°	154.43°	154.12°
	φ_4	162.89°	162.94°	162.89°	162.93°	162.95°
Region D	σ_1/k	- 1.82	- 2.04	- 2.35	- 3.26	- 5.95
	σ_2/k	0.0259	- 0.218	- 0.546	- 1.44	- 4.16
	J_1/k	- 2.02	- 2.67	- 3.61	- 6.32	- 14.47
	β	4.12	3.83	3.66	3.80	3.89
	θ	90°	90°	90°	90°	90°

TABLE 7 (For definition of regions, see Fig. 13)

$\nu = 0.125$ $V = 2.00\ c_P$ $\varphi_P = 150.00^\circ$ $\varphi_S = 160.89^\circ$ $\bar{\varphi} = 160.89^\circ$

		p_E $p_o = 1.91k$	$p_E < p_o < p_L$ $p_o = 2.13k$	p_L $p_o = 2.29k$	$p_L < p_o < p_H$ $p_o = 2.69k$	p_H $p_o = 5.76k$
Region A	σ_1/k	- 1.77	- 1.93	- 2.02	- 2.02	- 2.02
	σ_2/k	- 0.253	- 0.276	- 0.289	- 0.289	- 0.289
	J_1/k	- 2.28	- 2.48	- 2.60	- 2.60	- 2.60
	β	3.00	3.00	3.00	3.00	3.00
	θ	60.00°	60.00°	60.00°	60.00°	60.00°
	φ_1	Not Applicable	Not Applicable	158.16°	158.16°	158.16°
	φ_2	"	"	158.16°	159.80°	160.89°
Region B	σ_1/k	"	"	Not Applicable	- 2.21	- 5.16
	σ_2/k	"	"	"	- 0.499	- 3.44
	J_1/k	"	"	"	- 3.18	- 12.06
	β	"	"	"	2.89	3.00
	θ	"	"	"	66.59°	70.89°
	$\Delta\tau/k$	- 0.838	- 0.733	- 0.643	- 0.257	0
Region C	σ_1/k	Not Applicable	- 1.98	- 2.02	- 2.21	- 5.16
	σ_2/k	"	- 0.224	- 0.289	- 0.499	- 3.44
	J_1/k	"	- 2.48	- 2.60	- 3.18	- 12.06
	β	"	3.19	3.00	2.89	3.00
	θ	"	85.47°	81.79°	75.20°	70.89°
	φ_3	166.00°	164.73°	163.73°	162.00°	160.89°
	φ_4	166.00°	166.02°	166.01°	166.02°	166.04°
Region D	σ_1/k	- 1.91	- 2.13	- 2.29	- 2.69	- 5.76
	σ_2/k	- 0.115	- 0.348	- 0.507	- 0.902	- 3.99
	J_1/k	- 2.28	- 2.94	- 3.41	- 4.60	- 13.86
	β	3.55	3.43	3.39	3.44	3.52
	θ	90°	90°	90°	90°	90°

TABLE 8 (For definition of regions, see Fig. 13)

$\nu = 0.125$ $V = 4.00$ c_P $\varphi_P = 165.52^\circ$ $\varphi_S = 170.58^\circ$ $\bar{\varphi} = 170.58^\circ$

		p_E $p_o = 2.00k$	$p_E < p_o < p_L$ $p_o = 2.09k$	p_L $p_o = 2.16k$	$p_L < p_o < p_H$ $p_o = 2.30k$	p_H $p_o = 5.44k$
Region A	σ_1/k	- 1.96	- 2.00	- 2.02	- 2.02	- 2.02
	σ_2/k	- 0.280	- 0.286	- 0.289	- 0.289	- 0.289
	J_1/k	- 2.52	- 2.57	- 2.60	- 2.60	- 2.60
	β	3.00	3.00	3.00	3.00	3.00
	θ	75.52°	75.52°	75.52°	75.52°	75.52°
	φ_1	Not Applicable	Not Applicable	169.87°	169.87°	169.87°
	φ_2	"	"	169.87°	170.20°	170.58°
Region B	σ_1/k	"	"	Not Applicable	- 2.09	- 5.16
	σ_2/k	"	"	"	- 0.364	- 3.42
	J_1/k	"	"	"	- 2.81	- 12.00
	β	"	"	"	- 2.97	3.00
	θ	"	"	"	77.92°	80.60°
	$\Delta\tau/k$	- 0.430	- 0.360	- 0.304	- 0.160	0
Region C	σ_1/k	Not Applicable	- 2.01	- 2.02	- 2.09	- 5.16
	σ_2/k	"	- 0.273	- 0.289	- 0.364	- 3.42
	J_1/k	"	- 2.57	- 2.60	- 2.81	- 12.00
	β	"	3.04	3.00	2.97	3.00
	θ	"	87.55°	85.64°	83.24°	80.60°
	φ_3	171.96°	171.59°	171.31°	170.96°	170.58°
	φ_4	171.96°	171.96°	171.96°	171.96°	171.96°
Region D	σ_1/k	- 2.00	- 2.09	- 2.16	- 2.30	- 5.44
	σ_2/k	- 0.246	- 0.341	- 0.411	- 0.555	- 3.70
	J_1/k	- 2.52	- 2.80	- 3.01	- 3.44	- 12.87
	β	3.12	3.11	3.11	3.11	3.14
	θ	90°	90°	90°	90°	90°

TABLE 9 (For definition of regions, see Fig. 13)

$\nu = 0.25$ $V = 1.25\ c_2$ $\varphi_P = 126.87^\circ$ $\varphi_S = 152.49^\circ$ $\bar{\varphi} = 143.40^\circ$

		p_E $p_o = 2.50\ k$	$p_E < p_o < p_L$ $p_o = 2.61\ k$	p_L $p_o = 2.65\ k$	$p_L < p_o < p_H$ $p_o = 3.84\ k$	p_H $p_o = 10.5\ k$
Region A	σ_1/k	- 2.43	- 2.55	- 2.60	- 2.60	- 2.60
	σ_2/k	- 0.810	- 0.850	- 0.866	- 0.866	- 0.866
	J_1/k	- 4.05	- 4.25	- 4.33	- 4.33	- 4.33
	β	3.00	3.00	3.00	3.00	3.00
	θ	36.87°	36.87°	36.87°	36.87°	36.87°
	φ_1	Not Applicable	Not Applicable	139.39°	139.39°	139.39°
	φ_2	"	"	139.39°	142.51°	143.396°
Region B	σ_1/k	"	"	Not Applicable	- 3.51	- 10.1
	σ_2/k	"	"	"	- 1.81	- 8.35
	J_1/k	"	"	"	- 7.08	- 26.8
	β	"	"	"	2.82	3.00
	θ	"	"	"	47.46°	53.4°
	$\Delta\tau/k$	- 1.36	- 1.35	- 1.35	- 0.851	- 0.550
Region C	σ_1/k	Not Applicable	- 2.57	- 2.60	- 3.51	- 10.1
	σ_2/k	"	- 0.830	- 0.866	- 1.81	- 8.35
	J_1/k	"	- 4.25	- 4.33	- 7.08	- 26.8
	β	"	3.07	3.00	2.82	3.00
	θ	"	88.60°	88.11°	77.52°	71.6°
	φ_3	160.61°	160.05°	159.81°	156.12°	154.40°
	φ_4	160.61°	160.61°	160.59°	160.88°	161.01°
Region D	σ_1/k	- 2.50	- 2.61	- 2.65	- 3.84	- 10.5
	σ_2/k	- 0.735	- 0.851	- 0.897	- 2.03	- 8.69
	J_1/k	- 4.05	- 4.38	- 4.50	- 8.07	- 28.1
	β	3.28	3.21	3.19	3.74	4.12
	θ	90°	90°	90°	90°	90°

TABLE 10 (For definition of regions, see Fig. 13)

$\nu = 0.25$ $V = 1.50\ c_P$ $\varphi_P = 138.19^\circ$ $\varphi_S = 157.30^\circ$ $\bar{\varphi} = 150.20^\circ$

		p_E $p_o = 2.48k$	$p_E < p_o < p_L$ $p_o = 2.63k$	p_L $p_o = 2.69k$	$p_L < p_o < p_H$ $p_o = 4.31k$	p_H $p_o = 11.6k$
Region A	σ_1/k	- 2.39	- 2.54	- 2.60	- 2.60	- 2.60
	σ_2/k	- 0.797	- 0.846	- 0.866	- 0.866	- 0.866
	J_1/k	- 3.98	- 4.23	- 4.33	- 4.33	- 4.33
	β	3.00	3.00	3.00	3.00	3.00
	θ	48.19°	48.19°	48.19°	48.19°	48.19°
	φ_1	Not Applicable	Not Applicable	148.06°	148.06°	148.06°
	φ_2	"	"	148.06°	150.13°	150.20°
Region B	σ_1/k	"	"	Not Applicable	- 3.99	- 11.25
	σ_2/k	"	"	"	- 2.27	- 9.53
	J_1/k	"	"	"	- 8.51	- 30.31
	β	"	"	"	2.91	3.00
	θ	"	"	"	58.51°	60.21°
	$\Delta\tau/k$	- 1.13	- 1.09	- 1.07	- 0.523	- 0.425
Region C	σ_1/k	Not Applicable	- 2.56	- 2.60	- 3.99	- 11.25
	σ_2/k	"	- 0.820	- 0.866	- 2.27	- 9.53
	J_1/k	"	- 4.23	- 4.33	- 8.51	- 30.31
	β	"	3.09	3.00	2.91	3.00
	θ	"	87.61°	86.54°	76.21°	74.52°
	φ_3	162.95°	162.14°	161.78°	158.83°	158.43°
	φ_4	162.95°	162.95°	162.95°	163.09°	163.13°
Region D	σ_1/k	- 2.48	- 2.63	- 2.69	- 4.31	- 11.60
	σ_2/k	- 0.705	- 0.861	- 0.925	- 2.50	- 9.78
	J_1/k	- 3.98	- 4.43	- 4.61	- 9.47	- 30.00
	β	3.34	3.28	3.26	3.66	3.78
	θ	90°	90°	90°	90°	90°

TABLE 11 (For definition of regions, see Fig. 13)

$\nu = 0.25$ $V = 2.00\ c_P$ $\varphi_P = 150.00^\circ$ $\varphi_S = 163.22^\circ$ $\bar{\varphi} = 158.12^\circ$

		p_E $p_o = 2.52k$	$p_E < p_o < p_L$ $p_o = 2.61k$	p_L $p_o = 2.68k$	$p_L < p_o < p_H$ $p_o = 4.41k$	p_H $p_o = 11.4k$
Region A	σ_1/k	- 2.46	- 2.54	- 2.60	- 2.60	- 2.60
	σ_2/k	- 0.820	- 0.84	- 0.866	- 0.866	- 0.866
	J_1/k	- 4.10	- 4.23	- 4.33	- 4.33	- 4.33
	β	3.00	3.00	3.00	3.00	3.00
	θ	60.00°	60.00°	60.00°	60.00°	60.00°
	φ_1	Not Applicable	Not Applicable	157.22°	157.22°	157.22°
	φ_2	"	"	157.22°	158.11°	158.12°
Region B	σ_1/k	"	"	Not Applicable	- 4.20	- 11.16
	σ_2/k	"	"	"	- 2.47	- 9.43
	J_1/k	"	"	"	- 9.12	- 30.02
	β	"	"	"	- 2.96	3.00
	θ	"	"	"	67.46°	68.12°
	$\Delta\tau/k$	- 0.852	- 0.811	- 0.771	- 0.672	- 0.306
Region C	σ_1/k	Not Applicable	- 2.56	- 2.60	- 4.20	- 11.16
	σ_2/k	"	- 0.818	- 0.866	- 2.47	- 9.43
	J_1/k	"	- 4.23	- 4.33	- 9.12	- 30.02
	β	"	3.10	3.00	2.96	3.00
	θ	"	88.17°	86.44°	78.98°	78.32°
	φ_3	166.21°	165.75°	165.33°	163.79°	163.69°
	φ_4	166.21°	166.21°	166.21°	166.26°	166.27°
Region D	σ_1/k	- 2.52	- 2.61	- 2.68	- 4.41	- 11.38
	σ_2/k	- 0.759	- 0.848	- 0.923	- 2.63	- 9.60
	J_1/k	- 4.10	- 4.36	- 4.58	- 9.77	- 30.69
	β	3.22	3.20	3.19	3.40	3.44
	θ	90°	90°	90°	90°	90°

TABLE 12 (For definition of regions, see Fig. 13)

$\nu = 0.25$ $V = 4.00\ c_p$ $\varphi_p = 165.52^\circ$ $\varphi_s = 171.70^\circ$ $\bar{\varphi} = 169.26^\circ$

		p_E $p_o = 2.58k$	$p_E < p_o < p_L$ $p_o = 2.62k$	p_L $p_o = 2.63k$	$p_L < p_o < p_H$ $p_o = 4.36k$	p_H $p_o = 11.3k$
Region A	σ_1/k	- 2.56	- 2.59	- 2.60	- 2.60	- 2.60
	σ_2/k	- 0.854	- 0.863	- 0.866	- 0.866	- 0.866
	J_1/k	- 4.27	- 4.31	- 4.33	- 4.33	- 4.33
	β	3.00	3.00	3.00	3.00	3.00
	θ	75.52°	75.52°	75.52°	75.52°	75.52°
	φ_1	Not Applicable	Not Applicable	169.14°	169.14°	169.14°
	φ_2	"	"	169.14°	169.26°	169.26°
Region B	σ_1/k	"	"	Not Applicable	- 4.29	- 11.22
	σ_2/k	"	"	"	- 2.56	- 9.49
	J_1/k	"	"	"	- 9.40	- 30.2
	β	"	"	"	2.99	3.00
	θ	"	"	"	79.05°	79.26°
	$\Delta\tau/k$	- 0.431	- 0.388	- 0.371	- 0.337	- 0.147
Region C	σ_1/k	Not Applicable	- 2.59	- 2.60	- 4.29	- 11.22
	σ_2/k	"	- 0.859	- 0.866	- 2.56	- 9.49
	J_1/k	"	- 4.31	- 4.33	- 9.40	- 30.2
	β	"	3.01	3.00	2.99	3.00
	ϵ	"	88.49°	87.88°	84.35°	84.14°
	φ_3	172.27°	172.11°	172.05°	171.78°	171.76°
	φ_4	172.27°	172.27°	172.27°	172.27°	172.27°
Region D	σ_1/k	- 2.58	- 2.62	- 2.63	- 4.36	- 11.3
	σ_2/k	- 0.837	- 0.877	- 0.891	- 2.61	- 9.55
	J_1/k	- 4.27	- 4.39	- 4.43	- 9.61	- 30.42
	β	3.06	3.06	3.06	3.10	3.11
	θ	90°	90°	90°	90°	90°

TABLE 13 (For definition of regions, see Fig. 13)

$\nu = 0.35$ $V = 1.25\ c_P$ $\varphi_P = 126.87^\circ$ $\varphi_S = 157.40^\circ$ $\bar{\varphi} = 138.27^\circ$

		p_E $p_o = 3.48k$	$p_E < p_o < p_L$ $p_o = 4.43k$	p_L $p_o = 6.08k$	$p_L < p_o < p_H$ $p_o = 6.72k$	p_H $p_o = 11.4k$
Region A	σ_1/k	- 3.75	- 3.75	- 3.75	- 3.75	- 3.75
	σ_2/k	- 2.02	- 2.02	- 2.02	- 2.02	- 2.02
	J_1/k	- 7.79	- 7.79	- 7.79	- 7.79	- 7.79
	β	3.00	3.00	3.00	3.00	3.00
	θ	36.87°	36.87°	36.87°	36.87°	36.87°
	φ_1	136.87°	136.87°	136.87°	136.87°	136.87°
	φ_2	136.87°	137.55°	138.10°	138.18°	138.27°
Region B	σ_1/k	Not Applicable	- 4.57	- 6.08	- 6.69	- 11.31
	σ_2/k	"	- 2.84	- 4.35	- 4.97	- 9.58
	J_1/k	"	- 10.23	- 14.76	- 16.61	- 30.48
	β	"	2.94	2.93	2.94	2.99
	θ	"	42.49°	44.80°	45.79°	48.09°
	$\Delta\tau/k$	- 1.18	- 1.18	- 1.22	- 1.18	- 1.08
Region C	σ_1/k	Not Applicable	Not Applicable	Not Applicable	- 6.69	- 11.31
	σ_2/k	"	"	"	- 4.97	- 9.58
	J_1/k	"	"	"	- 16.61	- 30.48
	β	"	"	"	2.94	2.99
	θ	"	"	"	89.01°	86.71°
	φ_3	"	"	161.97°	161.67°	161.05°
	φ_4	"	"	161.97°	162.00°	162.11°
Region D	σ_1/k	- 3.48	- 4.43	- 6.08	- 6.72	- 11.39
	σ_2/k	- 2.29	- 2.98	- 4.35	- 4.98	- 9.63
	J_1/k	- 7.79	- 10.23	- 14.76	- 16.68	- 30.71
	β	2.06	2.47	2.93	3.02	3.24
	θ	90°	90°	90°	90°	90°

TABLE 14 (For definition of regions, see Fig. 13)

$\nu = 0.35 \qquad V = 1.50\ c_P \qquad \varphi_P = 138.19^\circ \qquad \varphi_S = 161.32^\circ \qquad \bar{\varphi} = 146.31^\circ$

		p_E $p_o = 3.64k$	$p_E < p_o < p_L$ $p_o = 4.47k$	p_L $p_o = 5.23k$	$p_L < p_o < p_H$ $p_o = 6.09k$	p_H $p_o = 11.2k$
Region A	σ_1/k	- 3.75	- 3.75	- 3.75	- 3.75	- 3.75
	σ_2/k	- 2.02	- 2.02	- 2.02	- 2.02	- 2.02
	J_1/k	- 7.79	- 7.79	- 7.79	- 7.79	- 7.79
	β	3.00	3.00	3.00	3.00	3.00
	θ	48.19°	48.19°	48.19°	48.19°	48.19°
	φ_1	145.69°	145.69°	145.69°	145.69°	145.69°
	φ_2	145.69°	146.01°	146.15°	146.25°	146.31°
Region B	σ_1/k	Not Applicable	- 4.51	- 5.23	- 6.06	- 11.16
	σ_2/k	"	- 2.79	- 3.51	- 4.34	- 9.43
	J_1/k	"	- 10.08	- 12.23	- 14.73	- 30.00
	β	"	2.97	2.96	2.96	2.99
	θ	"	50.85°	52.50°	53.98°	56.18°
	$\Delta\tau/k$	- 1.08	- 1.06	- 1.05	- 0.985	- 0.880
Region C	σ_1/k	Not Applicable	Not Applicable	Not Applicable	- 5.34	- 11.16
	σ_2/k	"	"	"	- 4.11	- 9.43
	J_1/k	"	"	"	- 14.05	- 30.00
	β	"	"	"	2.96	2.99
	θ	"	"	"	88.67°	86.46
	φ_3	"	"	164.33°	163.98°	163.48°
	φ_4	"	"	164.33°	164.33°	164.38°
Region D	σ_1/k	- 3.64	- 4.47	- 5.23	- 6.09	- 11.23
	σ_2/k	- 2.13	- 2.83	- 3.51	- 4.36	- 9.47
	J_1/k	- 7.79	- 10.08	- 12.23	- 14.31	- 30.21
	β	2.61	2.75	2.96	3.03	3.21
	θ	90°	90°	90°	90°	90°

TABLE 15 (For definition of regions, see Fig. 13)

$\nu = 0.35$ $V = 2.00\ c_p$ $\varphi_P = 150.00^\circ$ $\varphi_S = 166.10^\circ$ $\bar{\varphi} = 155.42^\circ$

		p_E $p_o = 3.72k$	$p_E < p_o < p_L$ $p_o = 4.34k$	p_L $p_o = 4.94k$	$p_L < p_o < p_H$ $p_o = 5.71k$	p_H $p_o = 11.8k$
Region A	σ_1/k	- 3.75	- 3.75	- 3.75	- 3.75	- 3.75
	σ_2/k	- 2.02	- 2.02	- 2.02	- 2.02	- 2.02
	J_1/k	- 7.79	- 7.79	- 7.79	- 7.79	- 7.79
	β	3.00	3.00	3.00	3.00	3.00
	θ	60°	60°	60°	60°	60°
	φ_1	155.21°	155.21°	155.21°	155.21°	155.21°
	φ_2	155.21°	155.30°	155.35°	155.39°	155.42°
Region B	σ_1/k	Not Applicable	- 4.36	- 4.94	- 5.69	- 11.75
	σ_2/k	"	- 2.63	- 3.20	- 3.96	- 10.02
	J_1/k	"	- 9.60	- 11.37	- 13.60	- 31.78
	β	"	2.99	2.99	2.98	3.00
	θ	"	61.52°	62.59°	63.58°	65.35°
	$\Delta\tau/k$	- 0.848	- 0.820	- 0.801	- 0.732	- 0.634
Region C	σ_1/k	Not Applicable	Not Applicable	Not Applicable	- 5.69	- 11.75
	σ_2/k	"	"	"	- 3.96	- 10.02
	J_1/k	"	"	"	- 13.60	- 31.78
	β	"	"	"	2.98	3.00
	θ	"	"	"	88.63°	86.85°
	φ_3	"	"	167.58°	167.33°	167.05°
	φ_4	"	"	167.58°	167.58°	167.59°
Region D	σ_1/k	- 3.72	- 4.34	- 4.94	- 5.71	- 11.80
	σ_2/k	- 2.06	- 2.64	- 3.20	- 3.97	- 10.05
	J_1/k	- 7.79	- 9.60	- 11.37	- 13.67	- 31.93
	β	2.87	2.95	2.99	3.05	3.14
	θ	90°	90°	90°	90°	90°

TABLE 16 (For definition of regions, see Fig. 13)

$\nu = 0.35$ $V = 4.00\ c_P$ $\varphi_P = 165.52^\circ$ $\varphi_S = 173.10^\circ$ $\bar{\varphi} = 167.99^\circ$

		p_E $p_o = 3.75k$	$p_E < p_o < p_L$ $p_o = 4.15k$	p_L $p_o = 5.08k$	$p_L < p_o < p_H$ $p_o = 6.82k$	p_H $p_o = 11.3k$
Region A	σ_1/k	- 3.75	- 3.75	- 3.75	- 3.75	- 3.75
	σ_2/k	- 2.02	- 2.02	- 2.02	- 2.02	- 2.02
	J_1/k	- 7.79	- 7.79	- 7.79	- 7.79	- 7.79
	β	3.00	3.00	3.00	3.00	3.00
	θ	75.52°	75.52°	75.52°	75.52°	75.52°
	φ_1	167.97°	167.97°	167.97°	167.97°	167.97°
	φ_2	167.97°	167.98°	167.98°	167.993°	167.994°
Region B	σ_1/k	Not Applicable	- 4.15	- 5.08	- 6.81	- 11.25
	σ_2/k	"	- 2.42	- 3.35	- 5.08	- 9.51
	J_1/k	"	- 9.00	-11.8	- 16.97	- 30.27
	β	"	3.00	3.00	3.00	2.99
	θ	"	76.03°	76.84°	77.57°	77.96°
	$\Delta\tau/k$	- 0.432	- 0.418	- 0.394	- 0.332	- 0.310
Region C	σ_1/k	Not Applicable	Not Applicable	Not Applicable	- 6.81	- 11.25
	σ_2/k	"	"	"	- 5.08	- 9.51
	J_1/k	"	"	"	- 16.97	- 30.27
	β	"	"	"	3.00	2.99
	θ	"	"	"	88.64°	88.24°
	φ_3	"	"	173.33°	173.25°	173.23°
	φ_4	"	"	173.33°	173.33°	173.33°
Region D	σ_1/k	- 3.75	- 4.15	- 5.08	- 6.82	- 11.26
	σ_2/k	- 2.03	- 2.42	- 3.35	- 5.09	- 9.52
	J_1/k	- 7.79	- 9.00	-11.8	- 17.01	- 30.32
	β	2.98	2.99	3.00	3.03	3.03
	θ	90°	90°	90°	90°	90°

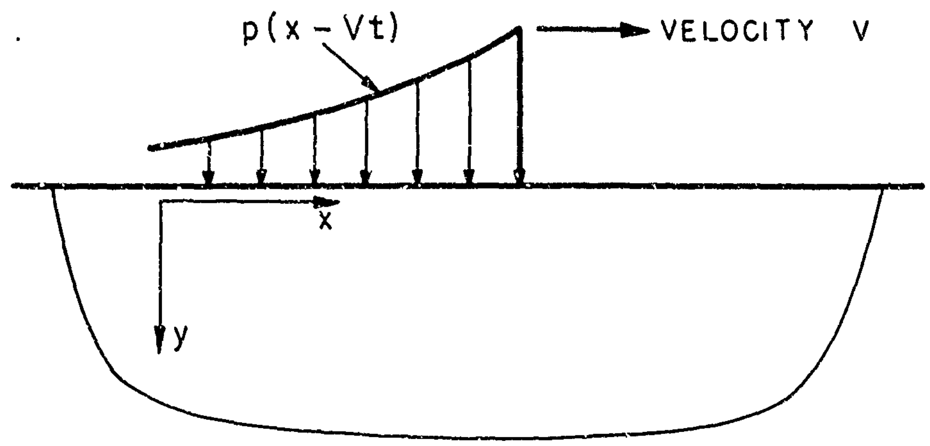


FIG. 1

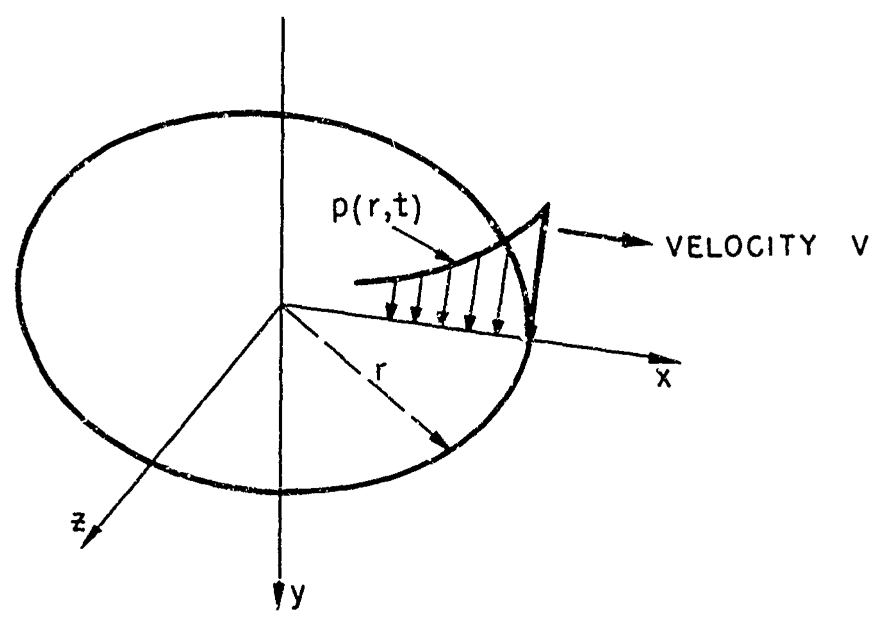


FIG. 2

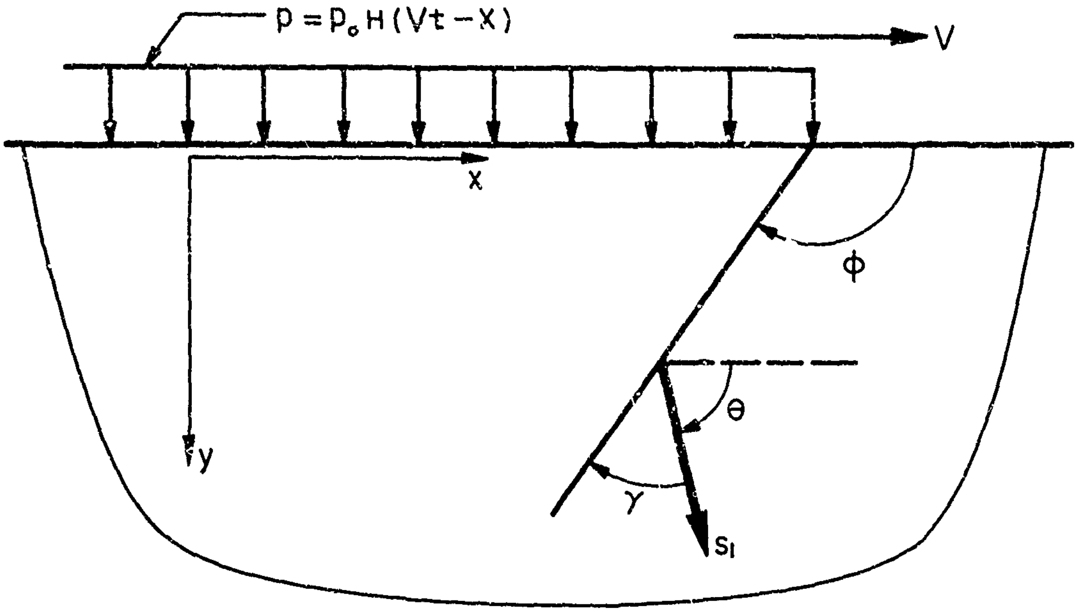


FIG. 3

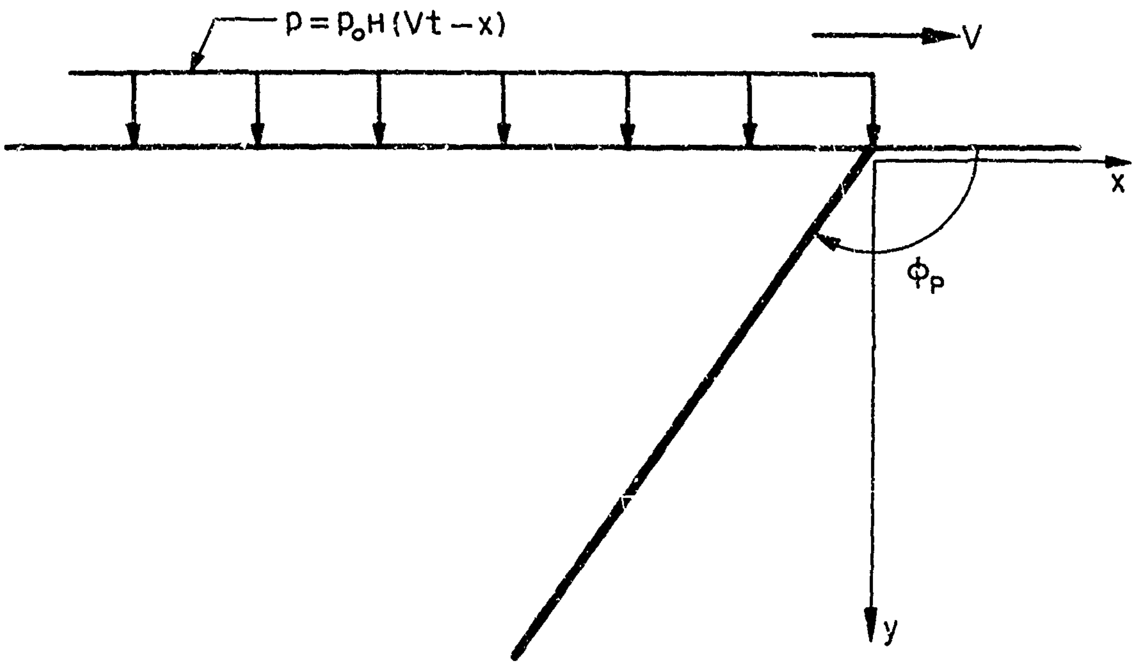
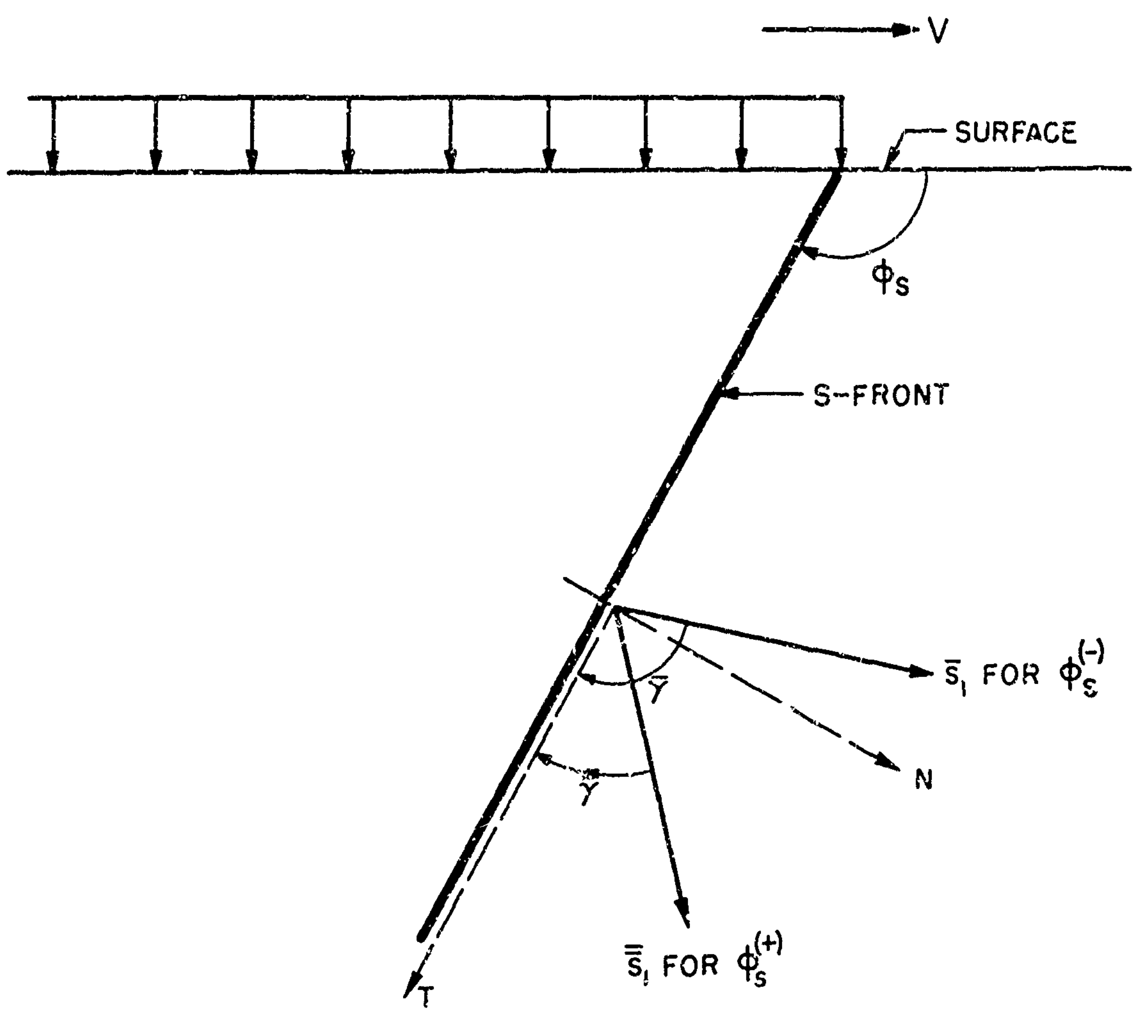


FIG. 4


V
SURFACE
ϕ_s
S-FRONT
$\bar{s}_1$ FOR $\phi_s^{(-)}$
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T

FIG. 5

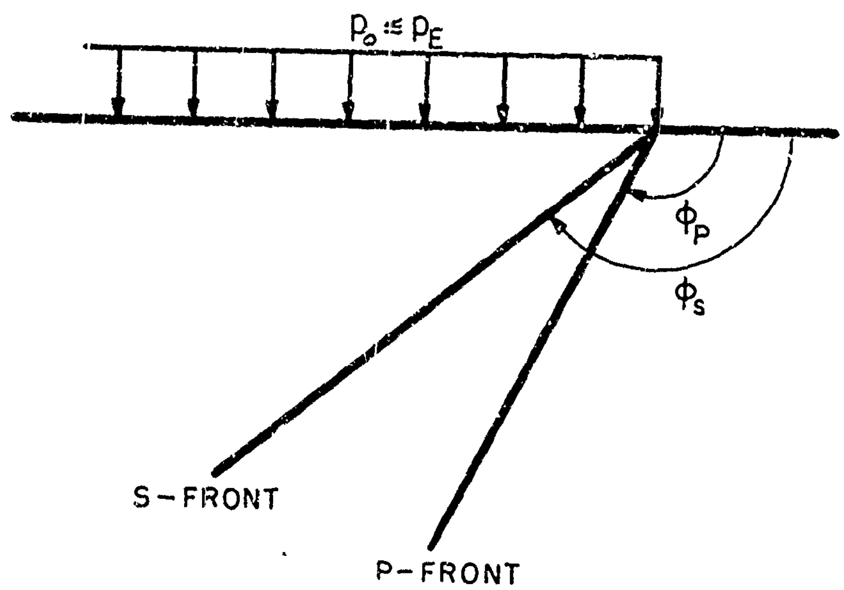


FIG. 6

CONFIGURATION OF ELASTIC SOLUTIONS

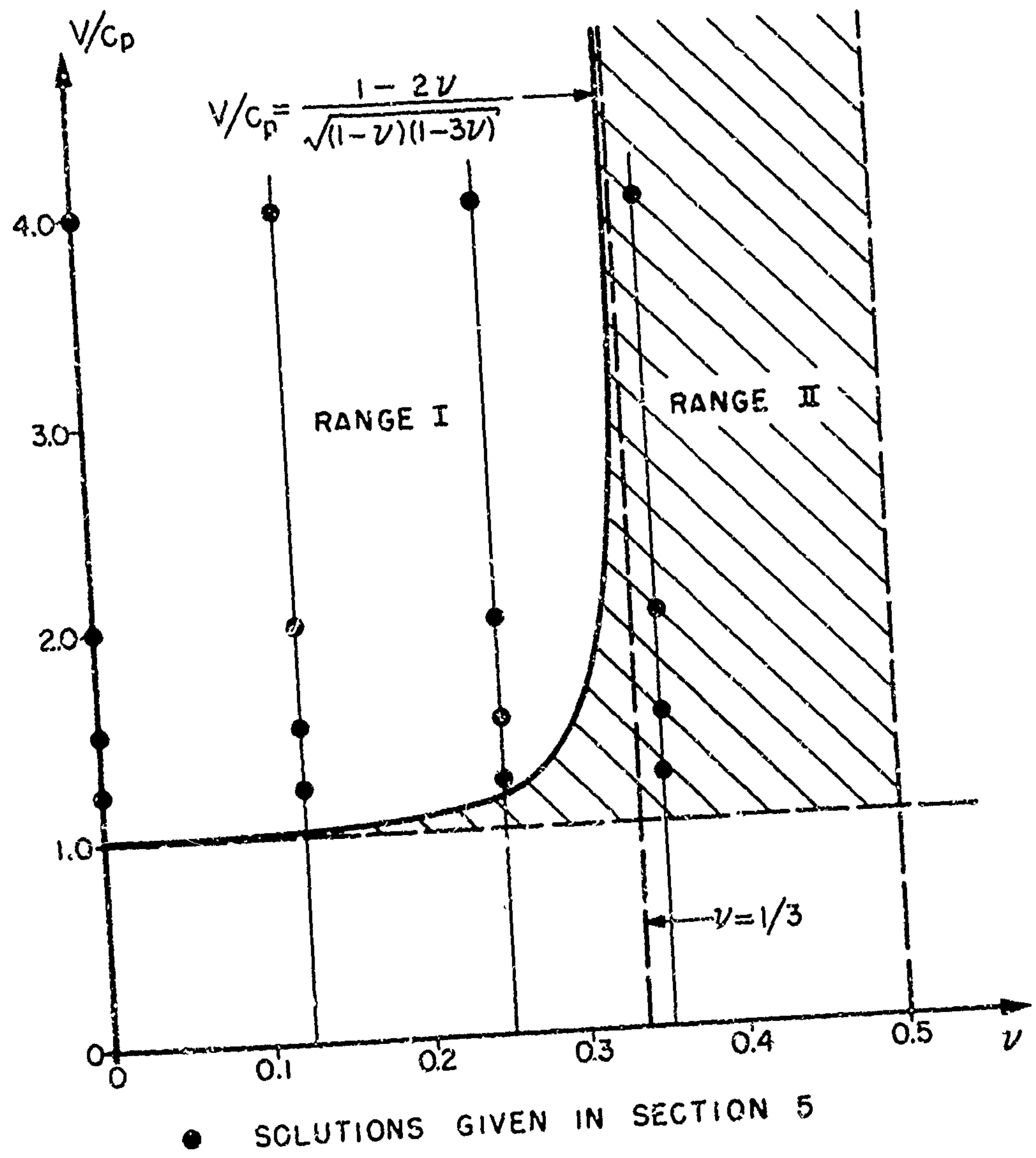

V/Cp
V/Cp = (1 − 2ν) / √((1−ν)(1−3ν))
4.0
3.0
2.0
1.0
0
RANGE I
RANGE II
ν=1/3
0
0.1
0.2
0.3
0.4
0.5
ν
● SOLUTIONS GIVEN IN SECTION 5

FIG. 7

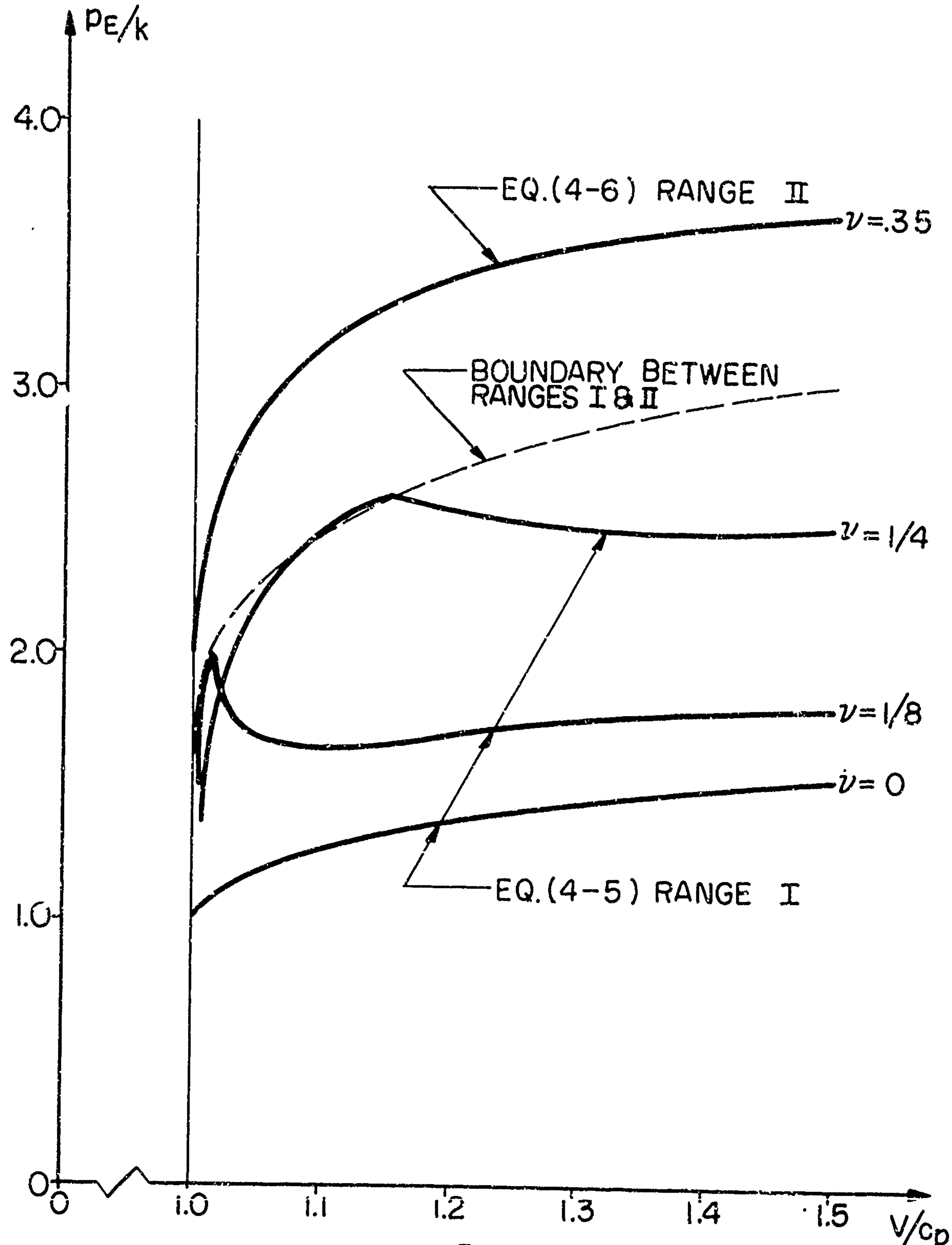


FIG 8

UPPER BOUND p_E/k TO ELASTIC SOLUTIONS

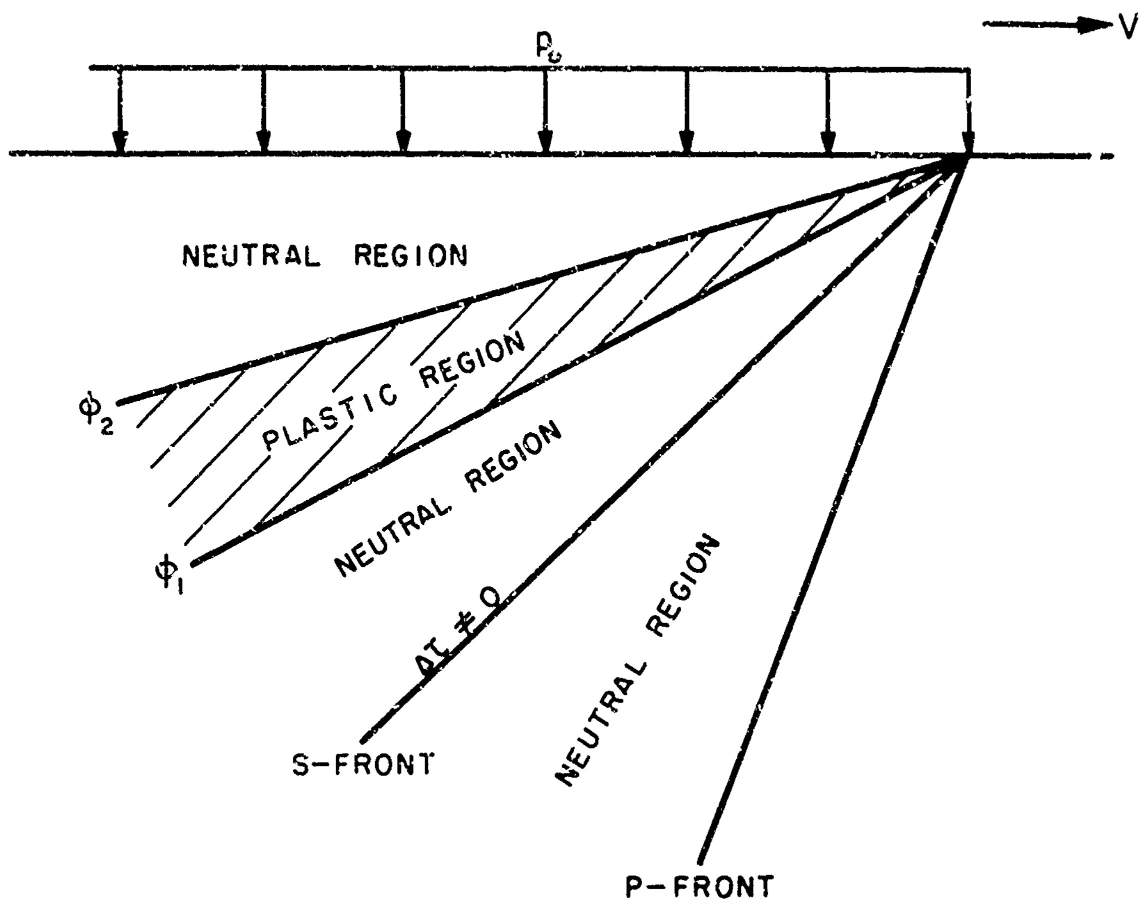

V
P_o
NEUTRAL REGION
PLASTIC REGION
ϕ_2
NEUTRAL REGION
ϕ_1
$\Delta\tau \neq 0$
NEUTRAL REGION
S-FRONT
P-FRONT

FIG. 9

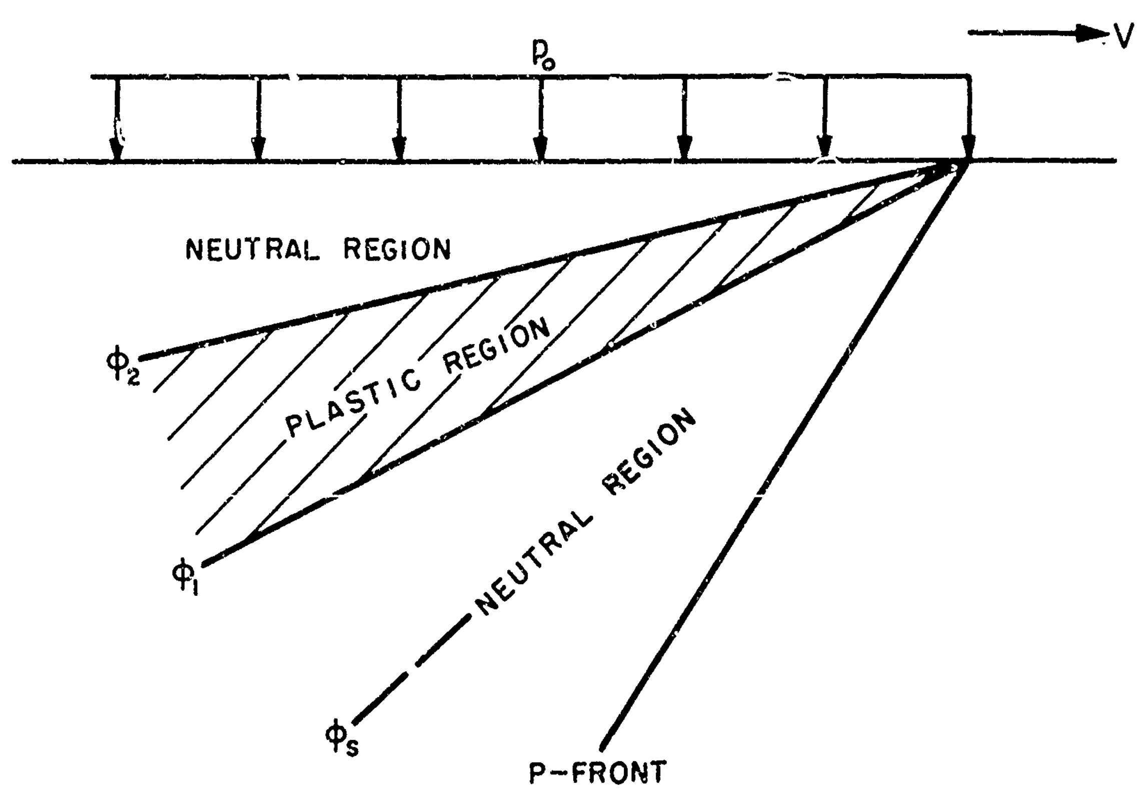

V
P_o
NEUTRAL REGION
PLASTIC REGION
ϕ_2
NEUTRAL REGION
ϕ_1
ϕ_s
P-FRONT

FIG 10a

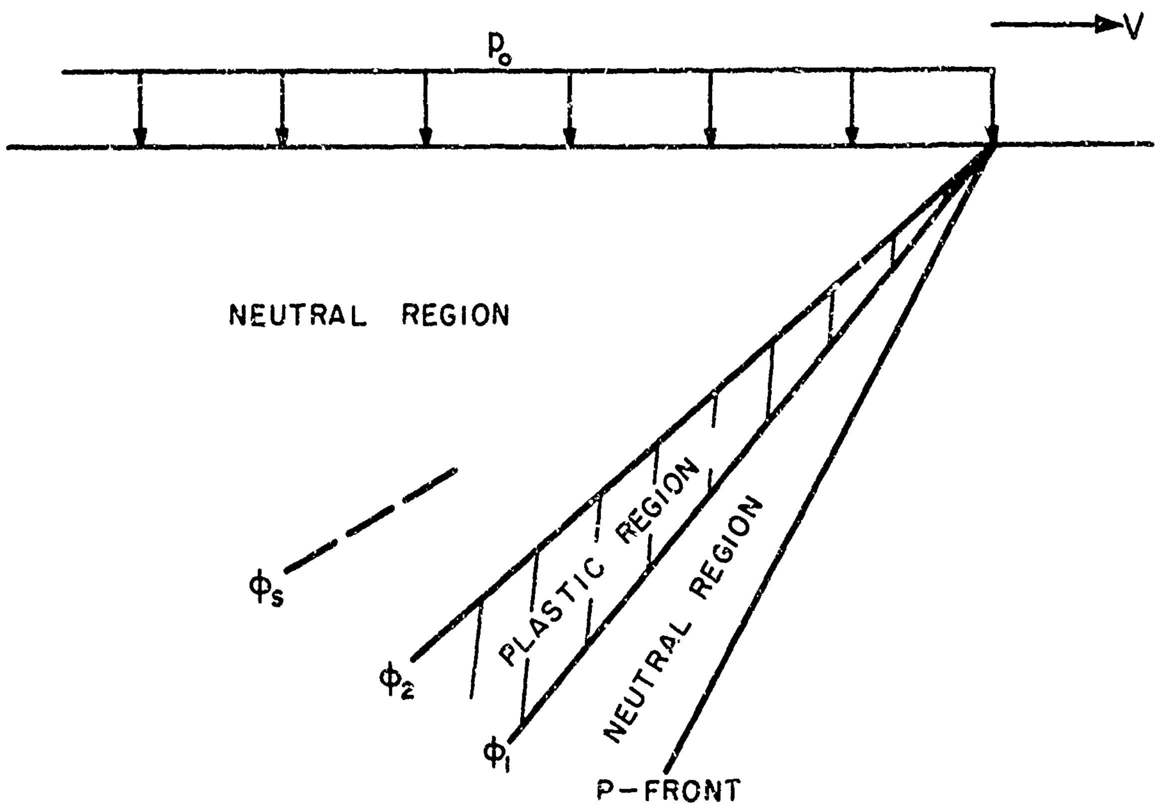


FIG 10b

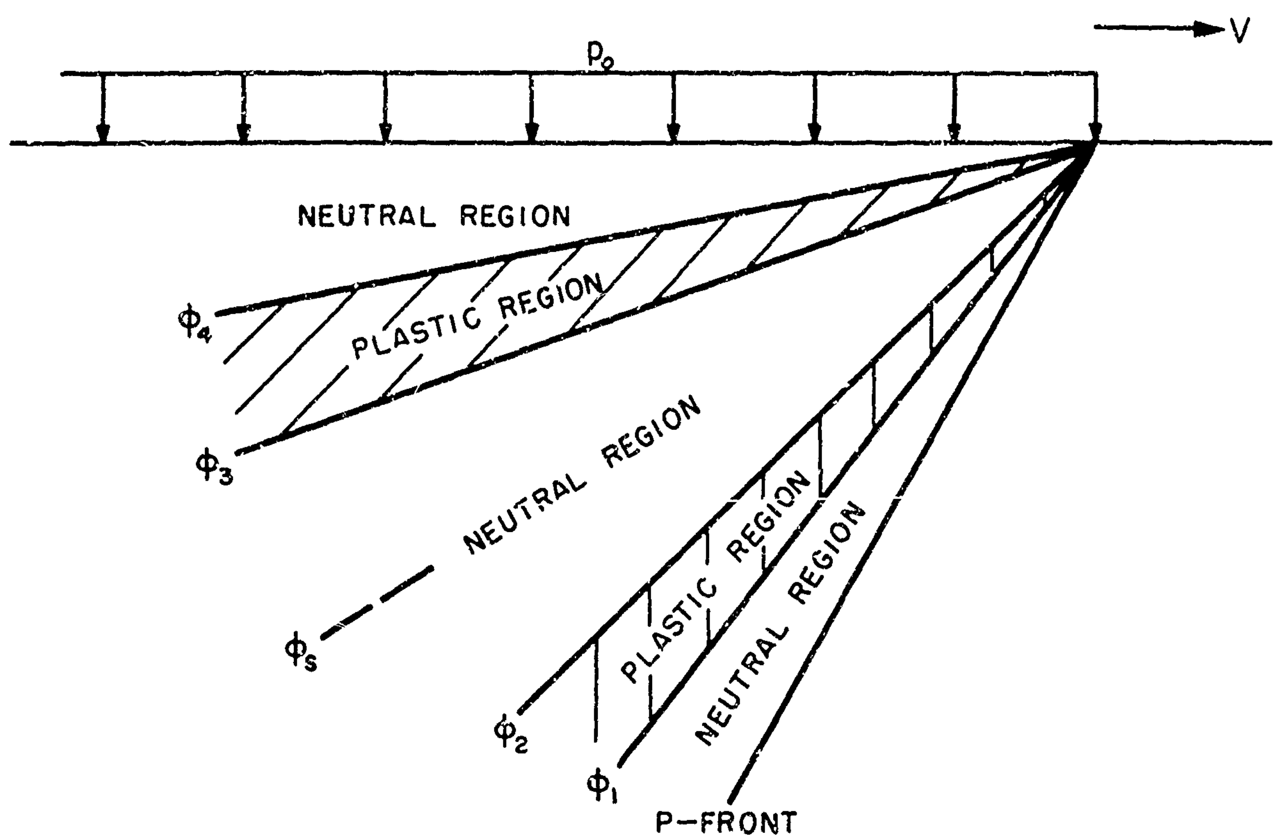


FIG 10C

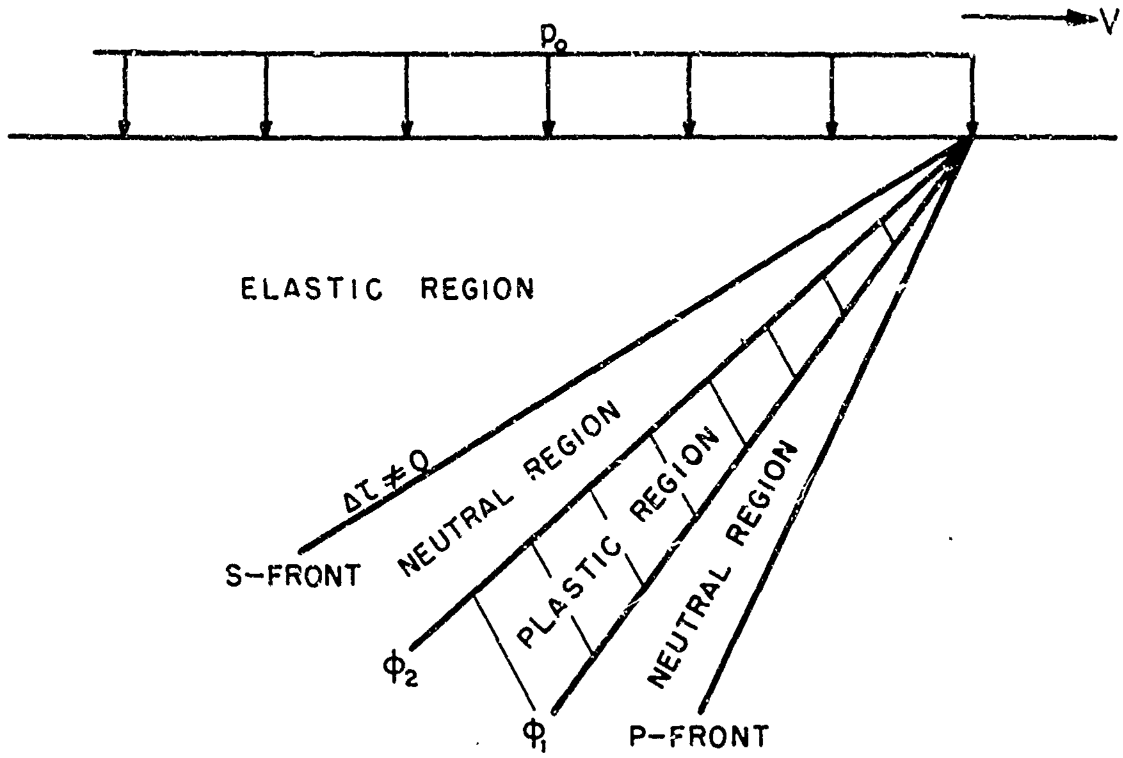

V
p_o
ELASTIC REGION
$\Delta\tau \neq 0$
NEUTRAL REGION
PLASTIC REGION
NEUTRAL REGION
S-FRONT
ϕ_2
ϕ_1
P-FRONT

FIG. 11

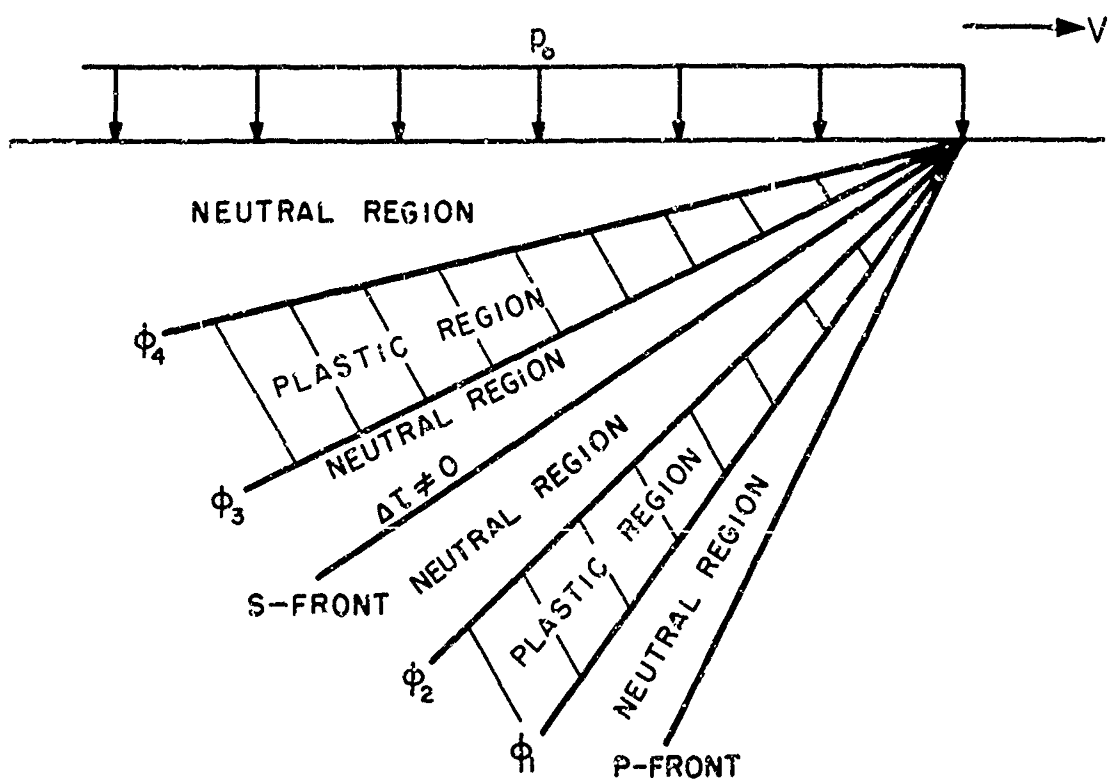

V
p_o
NEUTRAL REGION
ϕ_4
PLASTIC REGION
NEUTRAL REGION
ϕ_3
$\Delta\tau \neq 0$
NEUTRAL REGION
PLASTIC REGION
NEUTRAL REGION
S-FRONT
ϕ_2
ϕ_1
P-FRONT

FIG 12

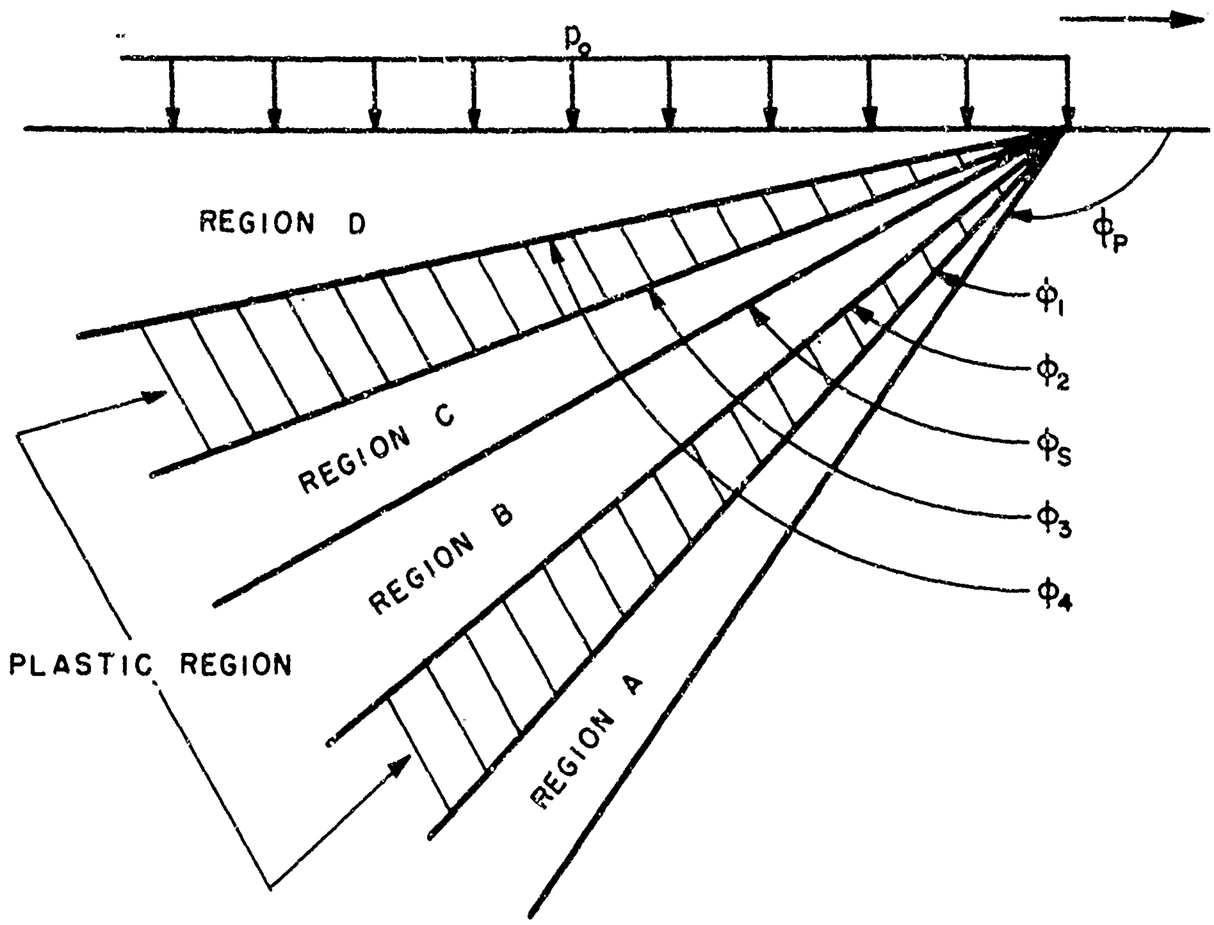


FIG 13

Referred to in Tables 1 to 16

Note that not all regions shown actually occur for all the cases of $\frac{p_o}{k}$.

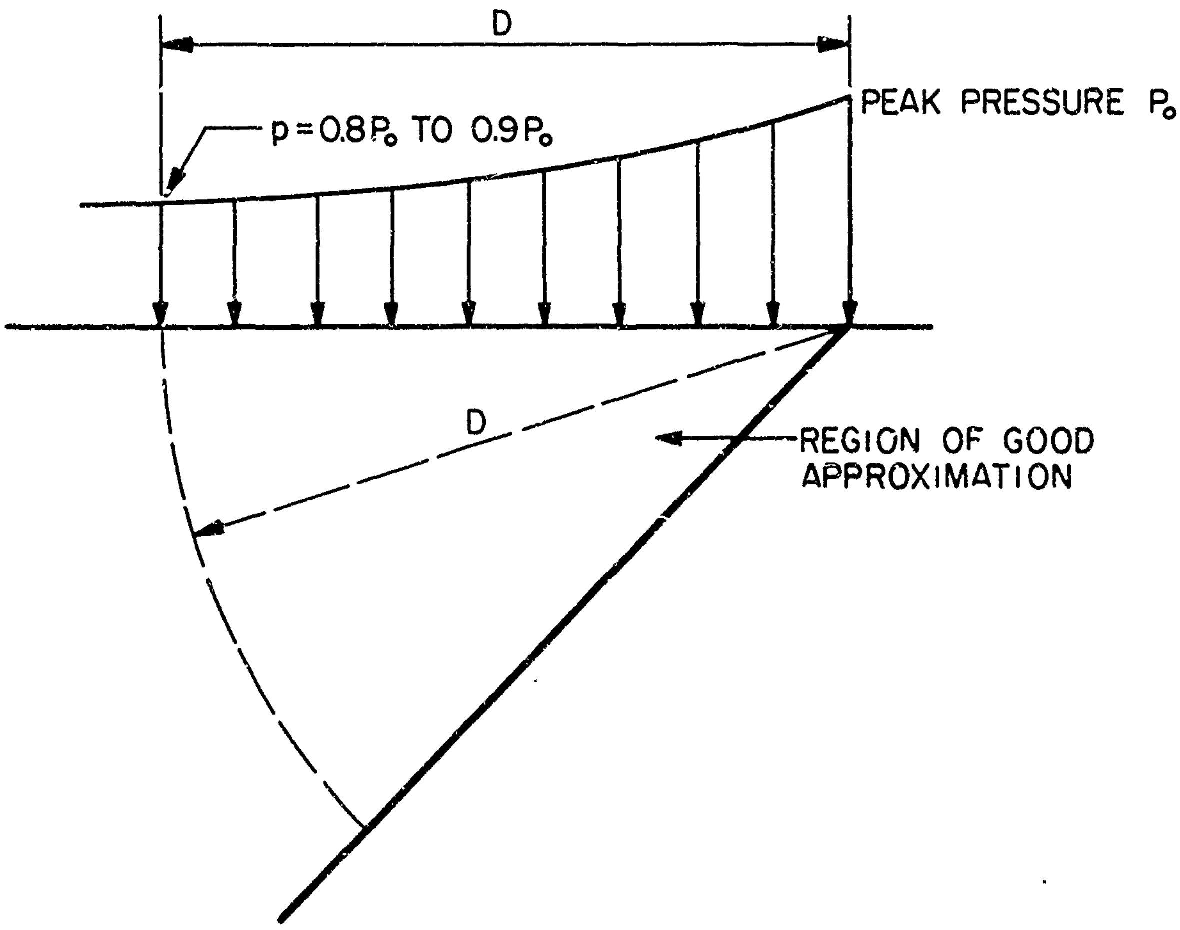
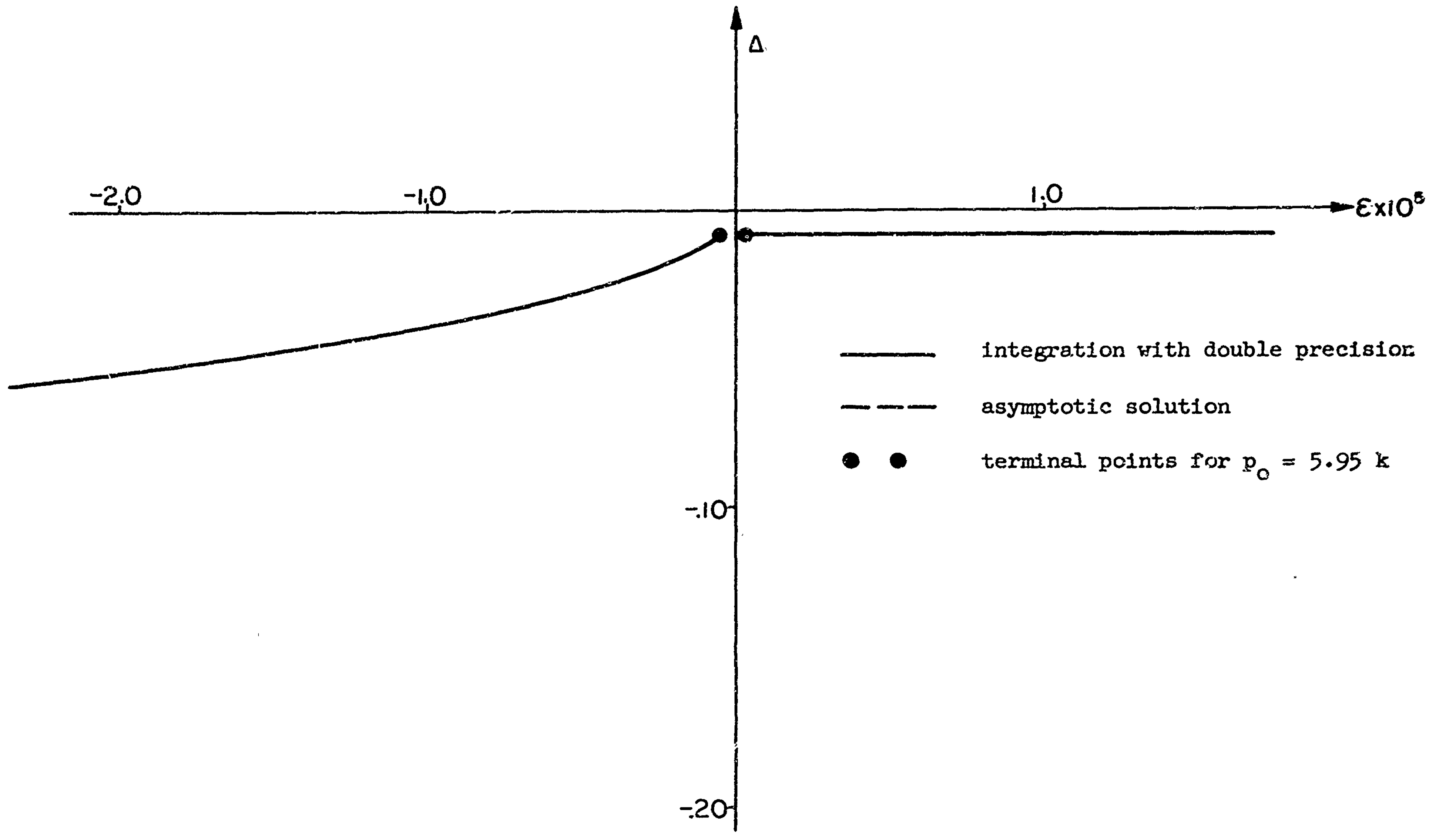
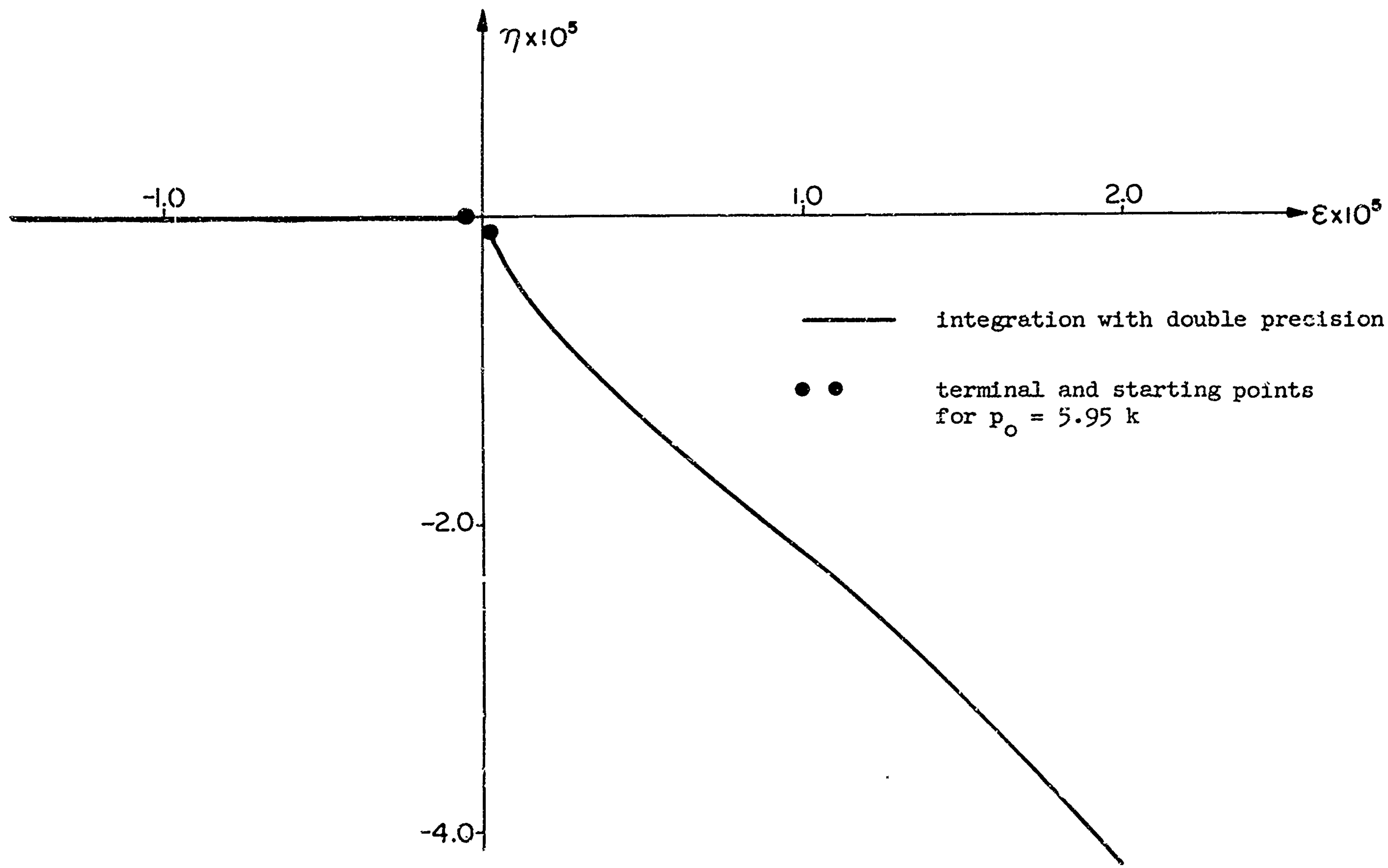

D
PEAK PRESSURE P_o
$p = 0.8P_o$ TO $0.9P_o$
D
REGION OF GOOD APPROXIMATION

FIG. 14


Δ
-2,0
-1,0
1,0
ε x 10⁵
-.10
-.20
integration with double precision
asymptotic solution
terminal points for p_o = 5.95 k

F I G . 15

$\nu = 0.125$, $V = 1.5\ C_p$



F I G . 16

$\nu = 0.125$, $V = 1.5\ C_p$

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The plane strain problem of a step load moving with uniform superseismic velocity $V > c_p$ on the surface of a half-space is considered for an elastic-plastic material obeying the von Mises yield condition.

Using dimensional analysis the governing quasi-linear partial differential equations are converted into ordinary nonlinear differential which are solved numerically using a digital computer. To overcome computing difficulties asymptotic solutions are derived in the vicinity of a singular point of the differential equations.

Numerical results are presented for a range of selected values of significant nondimensional parameters, i.e. of the surface load p_o/k, of Poisson's ratio ν and of the velocity ratio V/c_p.

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14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Stresses in Elastic-Plastic Half-Space Superseismic Step Load von Mises yield condition Wave Propagation						

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